VANISHING THEOREMS FOR COHERENT AUTOMORPHIC COHOMOLOGY

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Abstract. We consider the coherent cohomology of toroidal compactifications of locally symmetric varieties (such as Shimura varieties) with coefficients in the canonical and subcanonical extensions of automorphic vector bundles, and give explicit conditions for them to vanish in certain degrees. We also provide algorithms for determining all such degrees in practice.

Contents

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1. Introduction

The coherent cohomology of toroidal compactifications of locally symmetric varieties such as Shimura varieties, with coefficients in the so-called canonical and subcanonical extensions of automorphic (vector) bundles, has played important roles in the study of arithmetic properties of automorphic representations. (See [21] for an overview.) A fundamental question in such a study is to know in which degrees the cohomology groups are nonzero, or to rule out unnecessary complication by showing that all but some explicitly predictable degrees must be zero—this is the question of vanishing that we would like to address in this article.

When the locally symmetric varieties in question are compact, and when the coherent cohomology in question contributes to the Hodge graded pieces of the de Rham cohomology of automorphic local systems, the cohomology classes can be represented by harmonic forms which are directly related to automorphic forms, and there are rather general vanishing results due to Faltings in [13] and Vogan and Zuckerman in [51]. One of the most useful results is that, when the weight of the local system in question is regular, the corresponding de Rham cohomology is concentrated in the middle degree, and there is a similar result for the coherent cohomology contributing to the Hodge graded pieces of such de Rham cohomology. (Already in the compact case, there are coherent cohomology of automorphic bundles which might not contribute to any de Rham cohomology.)

However, when the locally symmetric varieties in question are not necessarily compact, our understanding is much less complete. The method of harmonic forms only gives information about the L^2 cohomology, which is in general not sufficient for the whole de Rham cohomology (or the compactly supported one, by duality), let alone the coherent cohomology that might not contribute to the Hodge graded pieces of any de Rham cohomology. (Here the coherent cohomology is defined over the toroidal compactifications as above, while the de Rham cohomology can also be defined over the toroidal compactifications using the de Rham complexes with integral connections with log poles along the boundary divisors.) Fortunately, thanks to Franke's results in [15], one can still study the (whole) de Rham cohomology using Eisenstein series and their residues, and it was shown by Li and Schwermer in [38] that, in the adelic setting, when the weight of the local system in question is regular, the corresponding de Rham cohomology vanishes below the middle degree, the compactly supported de Rham cohomology vanishes above the middle degree, and hence the interior cohomology, namely the image of the compactly supported cohomology in the usual cohomology, is concentrated in the middle degree. (Consequently, there are similar results for the coherent cohomology contributing to the Hodge graded pieces of such de Rham cohomology.)

Unfortunately, the techniques in [15] have not yet been generalized to also cover the case of coherent cohomology of canonical or subcanonical extensions of automorphic bundles of *noncohomological weights*, in the sense that the corresponding cohomology groups do not contribute to the Hodge graded pieces of the de Rham cohomology of any automorphic local system. (The representations of such noncohomological weights are characterized by having dual representations with irregular Harish–Chandra parameters.) To the best of our knowledge, it is still not known whether the coherent cohomology classes of such noncohomological weights are always represented by Eisenstein series and their residues. In this regard, the study in [37] of coherent cohomology of toroidal compactifications of PEL-type Shimura varieties in mixed characteristics provides nontrivial and new vanishing results for the coherent cohomology even in characteristic zero. In fact, the results such as [37, Thm. 8.13 and 8.23] (which are over the complex numbers) were new (although we were not fully aware of that at the time the results were published), and they still have not yet been reproved using techniques based on automorphic forms.

On the other hand, since the methods in [37] require the existence of good mixed characteristics models not only for the Shimura varieties and their toroidal compactifications (as in [32]), but also for the geometric families of abelian schemes and their toroidal compactifications (as in [31]) involved in the method, they have serious limitations. While we can imagine that the methods work very similarly for abelian-type Shimura varieties, we do not know how to extend them to more general cases. Note that there are Shimura varieties unrelated to exceptional groups which can still fail to be of abelian type—there are many such Shimura varieties, as explained in [41], associated with even orthogonal groups. Also, although we still know very little about Shimura varieties associated with exceptional groups, the theory feels incomplete and unsatisfactory if we cannot say anything about them.

Fortunately, the recent work by Suh (see [50]) allows us to extend the methods in [37] to arbitrary locally symmetric varieties considered in, e.g., [3] and [1], including even Shimura varieties associated with exceptional groups, and including even the noncongruence arithmetic group quotients of Hermitian symmetric domains. The key point is to replace the vanishing theorems in the first three sections of [37] (which were based on techniques in positive characteristics developed in [11], [25], [27], [12], and [43]) with a rather general vanishing theorem for mixed Hodge modules in [50] (which, however, is based on complex-analytic techniques in [44], which have no useful counterparts in positive characteristics yet).

While it might seem unsurprising that new vanishing theorems for automorphic cohomology are available once some new vanishing theorem for mixed Hodge modules as in [50] is known, we have been quite happily surprised by what (and how much) we could readily deduce from the latter, thanks to some pleasant facts in the combinatorics of root systems. For example, we have obtained a new method for reproving most of the Hermitian case of Li and Schwermer's vanishing theorem for the de Rham cohomology of local systems of regular weights, which is free of the consideration of automorphic forms, and hence is not reliant on the results of [15]. (Though we cannot say anything about the more general non-Hermitian cases also covered by their theorem.) Moreover, we have also obtained new vanishing results for coherent automorphic cohomology of low weights (not contributing to the Hodge graded pieces of the de Rham cohomology of local systems of regular weights), and we have found efficient algorithms for determining the degrees of vanishing in practice, in all possible (Hermitian) cases.

Here is an outline of the article. In Section 2, we review the necessary background materials for stating and proving the main results, concerning locally symmetric varieties and their toroidal and minimal compactifications, automorphic bundles and their canonical and subcanonical extensions, and the dual Bernstein–Gelfand–Gelfand (BGG) complexes. In Section 3, we describe the automorphic line bundles of what we call *positive parallel weights*, whose canonical extensions over toroidal compactifications associated with projective and smooth cone decompositions are *semiample* and satisfy a condition due to Esnault and Viehweg (so that the line bundles are, in particular, *nef* and *big*). We classify all such positive parallel weights, and give concrete descriptions of them in all cases. In Section 4, we state and prove most of our main results concerning the vanishing of coherent and de Rham cohomology, generalizing those in [36] and [37] (when specialized to the case over complex numbers), with byproducts giving new proofs of certain results in [33]. To help the reader understand our results, we also include some illustrative examples of low ranks. In Section 5, we explain our algorithms for determining the degrees of vanishing in all circumstances, and provide many explicit examples.

This article is written for people who would like to understand and use our vanishing results, and our judgement is that many of them will be number theorists or algebraic geometers rather than experienced representation theorists. (Some of the choices of conventions and notations might not be so natural for representation theorists, but they are made because of historical or practical reasons related to the geometric constructions or their number-theoretic applications.) Hence, while our arguments concerning roots and weights might be rather elementary and naive, we will still spell out most of the details, for the sake of clarity and readability. But we do not consider such efforts as merely expository—they are helpful for presenting our algorithms for determining the degrees of vanishing in all circumstances.

2. Background materials

2.1. **Locally symmetric varieties.** Let G be a reductive algebraic group over $\mathbb{Q}$ such that $\mathrm{G}(\mathbb{R})$ acts transitively on H, a finite disjoint union of Hermitian symmetric domains. Let h_0 be a fixed choice of a point of H, so that $\mathsf{H} = \mathrm{G}(\mathbb{R})h_0$, and let H_0 denote the connected component of h_0, which is a Hermitian symmetric domain by assumption. For expositional simplicity, suppose that the maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split subtorus $\overline{\mathrm{Z}}$ of the center Z of G is trivial (cf. [22, (1.1.7.3)]). (Otherwise, we shall assume instead that all representations we consider have trivial restrictions to $\overline{\mathrm{Z}}$; cf. [22, Rem. in (1.2)].)

Let G_0 denote the derived group of the connected component G° of the identity of G, which is a connected semisimple algebraic group over $\mathbb{Q}$ (see [47, Cor. 2.2.8 and 8.1.6(ii)]). Suppose $\mathsf{H}_0 \cong \mathrm{G}_0(\mathbb{R})/K_0$ for some maximal compact subgroup K_0 of $\mathrm{G}_0(\mathbb{R})$, which can be identified with the stabilizer of h_0 in $\mathrm{G}_0(\mathbb{R})$. Then there exists a parabolic subgroup P_0 of $\mathrm{G}_{0,\mathbb{C}} = \mathrm{G}_0 \underset{\mathbb{Q}}{\otimes} \mathbb{C}$, with a Levi subgroup M_0 which can be identified with the complexification of K_0 (via the identification of $\mathrm{G}_{0,\mathbb{C}}$ with the complexification of $\mathrm{G}_{0,\mathbb{R}} = \mathrm{G}_0 \underset{\mathbb{Q}}{\otimes} \mathbb{R}$), such that $K_0 = \mathrm{P}_0(\mathbb{C}) \cap \mathrm{G}_0(\mathbb{R})$ and the Borel embedding $\mathsf{H}_0 \hookrightarrow \mathsf{H}_0^\vee$ is given by $\mathrm{G}_0(\mathbb{R})/K_0 \to \mathrm{G}_0(\mathbb{C})/\mathrm{P}_0(\mathbb{C})$. (See, e.g., [23, Ch. VIII, Sec. 7], [1, Ch. III, Sec. 2.1] and [40, Sec. III.1].) Let us denote by $\widetilde{\mathrm{G}}_0$ the simply-connected covering of G_0, by $\widetilde{K}_0$ the preimage of K_0 in $\widetilde{\mathrm{G}}_0(\mathbb{R})$, by $\widetilde{\mathrm{P}}_0$ the preimage of P_0 in $\widetilde{\mathrm{G}}_{0,\mathbb{C}} = \widetilde{\mathrm{G}}_0 \underset{\mathbb{Q}}{\otimes} \mathbb{C}$, and by $\widetilde{\mathrm{M}}_0$ the preimage of M_0 in $\widetilde{\mathrm{P}}_0$. For simplicity, suppose that $\mathsf{H}_0 \hookrightarrow \mathsf{H}_0^\vee$ (necessarily uniquely) extends to a $\mathrm{G}(\mathbb{R})$-equivariant embedding $\mathsf{H} \cong \mathrm{G}(\mathbb{R})/K \hookrightarrow \mathsf{H}^\vee := \mathrm{G}(\mathbb{C})/\mathrm{P}(\mathbb{C})$, where P is the parabolic subgroup of $\mathrm{G}^\circ_\mathbb{C} = \mathrm{G}^\circ \underset{\mathbb{Q}}{\otimes} \mathbb{C}$ (uniquely) extending P_0, with a Levi subgroup M (uniquely) extending M_0, and where $K := \mathrm{P}(\mathbb{C}) \cap \mathrm{G}(\mathbb{R})$ extends K_0.

Suppose X is a complex analytic manifold such that there exist finitely many *neat* arithmetic subgroups Γ_i of $\mathrm{G}(\mathbb{Q})$ stabilizing H_0 and $g_i \in \mathrm{G}(\mathbb{R})$ such that $\mathsf{X} \cong \coprod_i \big((g_i \Gamma_i g_i^{-1}) \backslash (g_i \mathsf{H}_0)\big) \cong \coprod_i (\Gamma_i \backslash \mathsf{H}_0)$. By an explanation similar to that in [30, Sec. 2.5], based on [6, Thm. 5.1], this is the case when $\mathsf{X} \cong \mathrm{G}(\mathbb{Q})\backslash(\mathsf{H} \times \mathrm{G}(\mathbb{A}^\infty))/\mathcal{H}$ for some neat open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\mathbb{A}^\infty)$. (However, we also allow more general X.) By [3], X has the structure of a (possibly disconnected) quasi-projective variety, embedded in its *minimal compactification* $\mathsf{X}^{\min} \cong \coprod_i (\Gamma_i\backslash\mathsf{H}_0)^{\min}$, the latter being a projective normal variety. By [1] (see also [2]), for suitable choices of *projective* and *smooth* cone decompositions Σ_i's, the quasi-projective variety X admits a projective smooth toroidal compactification $\mathsf{X}^{\mathrm{tor}} \cong \coprod_i (\Gamma_i\backslash\mathsf{H}_0)^{\mathrm{tor}}_{\Sigma_i}$ whose boundary $\mathsf{D} := (\mathsf{X}^{\mathrm{tor}} - \mathsf{X})_{\mathrm{red}}$ (with its reduced structure) is a simple normal crossings divisor, which is equipped with a canonical proper surjective morphism $\oint : \mathsf{X}^{\mathrm{tor}} \to \mathsf{X}^{\min}$.

2.2. **Automorphic bundles and canonical extensions.** For each finite-dimensional algebraic representation W of P, in which case we write $W \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{P})$, we define a vector bundle $\underline{W}$ over H as the pullback under the embedding $\mathsf{H} \hookrightarrow \mathsf{H}^\vee = \mathrm{G}(\mathbb{C})/\mathrm{P}(\mathbb{C})$ of the analytification of the equivariant quotient $(\mathrm{G}_{\mathbb{C}} \times W)/\mathrm{P}$ over $\mathrm{G}_{\mathbb{C}}/\mathrm{P}$. For each i, the left action of $g_i\Gamma_i g_i^{-1}$ on $g_i\mathsf{H}_0$ lifts to an action on the restriction of $\underline{W}$ to $g_i\mathsf{H}_0$, and the disjoint union of such restrictions descends to a (holomorphic) *automorphic bundle* over X, which we still abusively denote by $\underline{W}$. Such a construction is functorial, exact, and compatible with tensor products and duals. We shall abusively denote the associated sheaves of sections by the same symbols.

For each finite-dimensional algebraic representation W of M, in which case we write $W \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{M})$, we view it as an object of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{P})$ via the canonical homomorphism $\mathrm{P} \to \mathrm{M}$, and define $\underline{W}$ over H and over X as above. By [42, Main Thm. 3.1], $\underline{W}$ admits a *canonical extension* $\underline{W}^{\mathrm{can}}$ over $\mathsf{X}^{\mathrm{tor}}$. Then we also define $\underline{W}^{\mathrm{sub}} := \underline{W}^{\mathrm{can}}(-\mathsf{D})$, where D is as above. Then it follows from GAGA [45] that $\underline{W}$, $\underline{W}^{\mathrm{can}}$, and $\underline{W}^{\mathrm{sub}}$ are all algebraic. By algebraizing extensions among them, the same assertion also holds for automorphic bundles and their canonical and subcanonical extensions associated with finite-dimensional algebraic representations of P.

For each finite-dimensional algebraic representation V of $\mathrm{G}_{\mathbb{C}}$, in which case we write $V \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{G}_{\mathbb{C}})$, we view it as an object of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{P})$ via the canonical homomorphism $\mathrm{P} \to \mathrm{G}_{\mathbb{C}}$, and define $\underline{V}$ over H and over X as above. Compared with the construction for $W \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{P})$, the action of $\mathrm{G}_{\mathbb{C}}$ (or rather its Lie algebra) on V allows us to equip $\underline{V}$ with an integrable connection $\nabla : \underline{V} \to \underline{V} \underset{\mathscr{O}_{\mathsf{X}}}{\otimes} \Omega^1_{\mathsf{X}/\mathbb{C}}$. As explained in [20, Sec. 4] (see also [40] and [21]), $(\underline{V}, \nabla)$ admits a canonical extension $(\underline{V}^{\mathrm{can}}, \nabla^{\mathrm{can}})$ over $\mathsf{X}^{\mathrm{tor}}$ in the sense of [10], where $\nabla^{\mathrm{can}} : \underline{V}^{\mathrm{can}} \to \underline{V}^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}}}{\otimes} \Omega^1_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D})$ is an integrable connection with log poles along D, with unipotent monodromy, by [1, Ch. III, Sec. 5, Main Thm. I and its proof] (and therefore with nilpotent residues, by [28, Sec. VI and VII]). We also define the subcanonical extension $(\underline{V}^{\mathrm{sub}}, \nabla^{\mathrm{sub}})$ by $\underline{V}^{\mathrm{sub}} := \underline{V}^{\mathrm{can}}(-\mathsf{D})$ and by setting ∇^{sub} to be the connection (also with log poles along D) canonically induced by ∇^{can}. Then we have the (log) de Rham complexes $\mathrm{DR}^\bullet(\underline{V}^{\mathrm{can}}) := (\underline{V}^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}}}{\otimes} \Omega^\bullet_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D}), \nabla^{\mathrm{can}})$ and $\mathrm{DR}^\bullet(\underline{V}^{\mathrm{sub}}) := (\underline{V}^{\mathrm{sub}} \underset{\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}}}{\otimes} \Omega^\bullet_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D}), \nabla^{\mathrm{sub}})$. These (log) de Rham complexes admit Hodge filtrations, which we denote by $\mathtt{F}$, given by the filtration on V induced by the action of the unipotent radical U of P, with associated Kodaira–Spencer complexes $\mathrm{Gr}^{\mathsf{F}}(\mathrm{DR}^{\bullet}(\underline{V}^{\mathrm{can}}))$ and $\mathrm{Gr}^{\mathsf{F}}(\mathrm{DR}^{\bullet}(\underline{V}^{\mathrm{sub}}))$ thanks to Griffiths transversality.

2.3. **Dual BGG complexes.** We shall denote by $\Phi_{\mathrm{G}_{\mathbb{C}}}$, Φ_{M}, etc the roots of $\mathrm{G}_{\mathbb{C}}$, M, etc, respectively; and by $\mathrm{X}_{\mathrm{G}_{\mathbb{C}}}$, X_{M}, etc the weights of $\mathrm{G}_{\mathbb{C}}$, M, etc, respectively. We shall fix the choice of a Borel subgroup B of $\mathrm{G}_{\mathbb{C}}^{\circ}$ such that $\mathrm{B} \subset \mathrm{P}$ and such that $\mathrm{B}_{\mathrm{M}} = \mathrm{B} \cap \mathrm{M}$ is a Borel subgroup of M, and fix a maximal torus T of B such that $\mathrm{T} \subset \mathrm{M} \subset \mathrm{P}$ is also a maximal torus of $\mathrm{G}_{\mathbb{C}}^{\circ}$. Then the choice of B determines the subsets of positive roots $\Phi_{\mathrm{G}_{\mathbb{C}}}^{+}$ and Φ_{M}^{+}, and of dominant weights $\mathrm{X}_{\mathrm{G}_{\mathbb{C}}}^{+}$ and $\mathrm{X}_{\mathrm{M}}^{+}$.

When W is an irreducible representation of highest weight $\nu \in \mathrm{X}_{\mathrm{M}}^{+}$, we write $W = W_{\nu}$, $\underline{W} = \underline{W}_{\nu}$, etc. Similarly, when G is connected and V is an irreducible representation of highest weight $\mu \in \mathrm{X}_{\mathrm{G}_{\mathbb{C}}}^{+}$, we write $V = V_{\mu}$, $\underline{V} = \underline{V}_{\mu}$, etc. When G is not connected, we will abusively denote by $V_{[\mu]}$ any irreducible representation of $\mathrm{G}_{\mathbb{C}}$ whose restriction to $\mathrm{G}_{\mathbb{C}}^{\circ}$ decomposes into a sum of irreducible representations $V_{\mu'}$, for all μ' in some multiset $[\mu]$ of dominant weights of $\mathrm{G}_{\mathbb{C}}^{\circ}$. The justification for this is that the geometric structures of the resulted $(\underline{V}_{[\mu]}, \nabla)$ and their canonical and subcanonical extensions only depend on the weights μ' in $[\mu]$, but not on the structure of $V_{[\mu]}$ as a representation of $\mathrm{G}_{\mathbb{C}}$. This terminology is not ideal, but suffices in many naturally occurring cases such as representations of orthogonal groups.

Definition 2.1. *We say that a root $\alpha \in \Phi_{\mathrm{G}_{\mathbb{C}}}$ is* **compact** *if $\alpha \in \Phi_{\mathrm{M}}$; otherwise we say it is* **noncompact**. *We shall denote the set of noncompact roots by $\Phi_{\mathrm{G}_{\mathbb{C}}}^{\mathrm{M}}$, and denote the positive noncompact roots by $\Phi_{\mathrm{G}_{\mathbb{C}}}^{\mathrm{M},+}$. We extend these notions and notations to the corresponding coroots in the obvious ways.*

As usual, let $\rho_{\mathrm{G}_{\mathbb{C}}} := \frac{1}{2} \sum\limits_{\mu \in \Phi_{\mathrm{G}_{\mathbb{C}}}^{+}} \mu$ and $\rho_{\mathrm{M}} := \frac{1}{2} \sum\limits_{\nu \in \Phi_{\mathrm{M}}^{+}} \nu$ denote the half-sums of positive roots, and let $\rho^{\mathrm{M}} := \rho_{\mathrm{G}_{\mathbb{C}}} - \rho_{\mathrm{M}}$. Let U denote (as above) the unipotent radical of P. Let $\mathfrak{g}$ (resp. $\mathfrak{p}$, resp. $\mathfrak{u}$) denote the Lie algebra of $\mathrm{G}_{\mathbb{C}}$ (resp. P, resp. U). Essentially by definition, $\mathfrak{u}$ is dual to $\mathfrak{g}/\mathfrak{p}$ as representations of M, and the weight of the top exterior power $\wedge^{\mathrm{top}} \mathfrak{u}$ is $2\rho^{\mathrm{M}} = \sum\limits_{\alpha \in \Phi_{\mathrm{G}_{\mathbb{C}}}^{\mathrm{M},+}} \alpha$. Then, for $d := \dim_{\mathbb{C}}(\mathsf{X}) = \dim_{\mathbb{C}}(\mathsf{H})$, we have $\Omega_{\mathsf{X}/\mathbb{C}}^{d} = \wedge^{\mathrm{top}} \Omega_{\mathsf{X}/\mathbb{C}}^{1} \cong \underline{W}_{2\rho^{\mathrm{M}}}$, $\Omega_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}^{d}(\log \mathsf{D}) \cong \underline{W}_{2\rho^{\mathrm{M}}}^{\mathrm{can}}$, and $\Omega_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}^{d} \cong \underline{W}_{2\rho^{\mathrm{M}}}^{\mathrm{sub}}$. Let $\mathrm{W}_{\mathrm{G}_{\mathbb{C}}}$ and W_{M} denote the Weyl groups of $\mathrm{G}_{\mathbb{C}}$ and M with respect to the common maximal torus T, which allows us to identify W_{M} as a subgroup of $\mathrm{W}_{\mathrm{G}_{\mathbb{C}}}$. In addition to the natural action of $\mathrm{W}_{\mathrm{G}_{\mathbb{C}}}$ on $\mathrm{X}_{\mathrm{G}_{\mathbb{C}}}$, there is also the dot action $w \cdot \mu = w(\mu + \rho_{\mathrm{G}_{\mathbb{C}}}) - \rho_{\mathrm{G}_{\mathbb{C}}}$, for all $w \in \mathrm{W}_{\mathrm{G}_{\mathbb{C}}}$ and $\mu \in \mathrm{X}_{\mathrm{G}_{\mathbb{C}}}$. Let W^{M} denote the subset of $\mathrm{W}_{\mathrm{G}_{\mathbb{C}}}$ consisting of elements w such that $w(\mathrm{X}_{\mathrm{G}_{\mathbb{C}}}^{+}) \subset \mathrm{X}_{\mathrm{M}}^{+}$.

Lemma 2.2. *For every $\alpha \in \Phi_{\mathrm{M}}$, we have $(\rho^{\mathrm{M}}, \alpha^{\vee}) = 0$.*

Proof. This is because $(\rho_{\mathrm{G}_{\mathbb{C}}}, \alpha^{\vee}) = 1 = (\rho_{\mathrm{M}}, \alpha^{\vee})$ for every simple α in Φ_{M}^{+}. □

Lemma 2.3. *For every $\alpha \in \Phi_{\mathrm{G}_{\mathbb{C}}}^{\mathrm{M},+}$, we have $(\rho^{\mathrm{M}}, \alpha^{\vee}) > 0$.*

Proof. We may and we shall replace $\mathrm{G}_{\mathbb{C}}$ with the $\mathbb{C}$-simple factors of $\widetilde{\mathrm{G}}_{0,\mathbb{C}}$, and assume that there is a unique simple $\alpha_0 \in \Phi_{\mathrm{G}_{\mathbb{C}}}^{\mathrm{M},+}$ (because the assertion is trivial when $\mathrm{M} = \mathrm{P} = \mathrm{G}_{\mathbb{C}}$). If $\alpha \in \Phi_{\mathrm{G}_{\mathbb{C}}}^{\mathrm{M},+}$, then $\alpha^{\vee}$ is the sum of some positive compact coroots and $r\alpha_0^{\vee}$ for some integer $r \geq 1$. On the other hand, while $2\rho^{\mathrm{M}} = \sum\limits_{\alpha \in \Phi_{\mathrm{G}_{\mathbb{C}}}^{\mathrm{M},+}} \alpha$

is the weight of the top exterior power $\wedge^{\mathrm{top}} \mathfrak{u}$, it is a positive multiple $s\varpi_0$ of the fundamental weight ϖ_0 (which is characterized by the property that $(\varpi_0, \alpha_0^\vee) = 1$ and $(\varpi_0, \alpha^\vee) = 0$ for all simple $\alpha \in \Phi_{\mathrm{M}}^+$). Therefore, by Lemma 2.2, we have $(\rho^{\mathrm{M}}, \alpha^\vee) = r(\rho^{\mathrm{M}}, \alpha_0^\vee) = \frac{1}{2} rs > 0$, as desired. □

Proposition 2.4 (Faltings)**.** *For each irreducible representation $V_{[\mu]}$ of $\mathrm{G}_{\mathbb{C}}$, and for $? = \mathrm{can}$ or sub, there is an $\mathtt{F}$-filtered complex $\mathrm{BGG}^\bullet((\underline{V}_{[\mu]}^\vee)^?)$, with trivial differentials on $\mathtt{F}$-graded pieces, such that*

$$\mathrm{Gr}_{\mathtt{F}}\big(\mathrm{BGG}^a((\underline{V}_{[\mu]}^\vee)^?)\big) \cong \mathop{\oplus}_{w \in \mathrm{W}^{\mathrm{M}},\, l(w)=a} \Big(\mathop{\oplus}_{\mu' \in [\mu]} (\underline{W}_{w\cdot\mu'}^\vee)^? \Big)$$

as $\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}}$-modules, together with a canonical quasi-isomorphic embedding

$$\mathrm{Gr}_{\mathtt{F}}\big(\mathrm{BGG}^\bullet((\underline{V}_{[\mu]}^\vee)^?)\big) \hookrightarrow \mathrm{Gr}_{\mathtt{F}}\big(\mathrm{DR}^\bullet((\underline{V}_{[\mu]}^\vee)^?)\big)$$

(*of complexes of $\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}}$-modules*) *between $\mathtt{F}$-graded pieces.*

Proof. This follows from the construction of dual Bernstein–Gelfand–Gelfand (BGG) complexes in [13, Sec. 3 and 7]. (See also [5] and [14, Ch. VI, Sec. 5].) □

Corollary 2.5. *For each irreducible representation $V_{[\mu]}$ of $\mathrm{G}_{\mathbb{C}}$, and for $? = \mathrm{can}$ or sub, we have a decomposition*

$$H^i\Big(\mathsf{X}^{\mathrm{tor}}, \mathrm{Gr}_{\mathtt{F}}\big(\mathrm{DR}^\bullet((\underline{V}_{[\mu]}^\vee)^?)\big)\Big) \cong \mathop{\oplus}_{w \in \mathrm{W}^{\mathrm{M}}, l(w)=a} \Big(\mathop{\oplus}_{\mu' \in [\mu]} H^{i-l(w)}\big(\mathsf{X}^{\mathrm{tor}}, (\underline{W}_{w\cdot\mu'}^\vee)^?\big)\Big)$$

whose left-hand side is the so-called Hodge cohomology (*giving the E_1 page of the Hodge spectral sequence for the de Rham cohomology $H^i(\mathsf{X}^{\mathrm{tor}}, \mathrm{DR}^\bullet((\underline{V}_{[\mu]}^\vee)^?))$*) *and whose right-hand side is a direct sum of coherent cohomology.*

Proof. This is an immediate consequence of Proposition 2.4. □

Corollary 2.5 provides the justification for the following:

Definition 2.6. *We say that $\nu \in \mathrm{X}_{\mathrm{M}}^+$ is* **cohomological** (*for the de Rham and Hodge cohomology*) *if there exist some* (*necessarily unique*) *$\mu = \mu(\nu) \in \mathrm{X}_{\mathrm{G}_{\mathbb{C}}}^+$ and $w = w(\nu) \in \mathrm{W}^{\mathrm{M}}$ such that $W_\nu \cong W_{w\cdot\mu}^\vee$.*

3. Positive parallel weights

3.1. Ampleness.

Definition 3.1. *We say that $\nu \in \mathrm{X}_{\mathrm{M}}^+$ is* **positive parallel** *if W_ν is one-dimensional and if, for each $\mathbb{Q}$-simple factor of $\widetilde{\mathrm{G}}_0$ that is noncompact at ∞, the pullbacks of ν and ρ^{M} to the corresponding factor of $\mathrm{X}_{\widetilde{\mathrm{M}}_0}^+$ are equal up to multiplication by a positive* (*rational*) *number.*

Lemma 3.2. *If $\nu \in \mathrm{X}_{\mathrm{M}}^+$ is positive parallel as in Definition 3.1, then the automorphic bundle $\underline{W}_\nu$ over X is an ample line bundle, and the canonical extension $\underline{W}_\nu^{\mathrm{can}}$ over $\mathsf{X}^{\mathrm{tor}}$ is a semiample line bundle, and there exists some integer $N \geq 1$ such that $\underline{W}_{N\nu}^{\mathrm{can}} \cong (\underline{W}_\nu^{\mathrm{can}})^{\otimes N}$ descends to an ample line bundle $\omega_{N\nu}$ over $\mathsf{X}^{\mathrm{min}}$.*

Proof. We may and we shall replace X with its finitely many connected components $(g_i\Gamma_i g_i^{-1})\backslash(g_i\mathsf{H}_0) \cong \Gamma_i\backslash\mathsf{H}_0$, replace G with $\widetilde{\mathrm{G}}_0$, replace H with H_0, and replace each arithmetic subgroup Γ_i of $\mathrm{G}(\mathbb{Q})$ with a neat finite index normal subgroup of its preimage in $\widetilde{\mathrm{G}}_0(\mathbb{Q})$. Accordingly, we shall replace $\mathsf{X}^{\mathrm{min}}$ and $\mathsf{X}^{\mathrm{tor}}$ with $(\Gamma_i\backslash\mathsf{H})^{\mathrm{min}}$

and $(\Gamma_i\backslash\mathsf{H})^{\mathrm{tor}}_{\Sigma_i}$, respectively, and replace each Σ_i with a projective and smooth refinement. (By Zariski's main theorem, for each finite index normal subgroup Γ'_i of Γ_i, the canonical morphism $(\Gamma'_i\backslash\mathsf{H})^{\mathrm{min}} \to (\Gamma_i\backslash\mathsf{H})^{\mathrm{min}}$ between projective normal varieties is finite and induces an isomorphism $(\Gamma'_i\backslash\Gamma_i)\backslash(\Gamma'_i\backslash\mathsf{H})^{\mathrm{min}} \xrightarrow{\sim} (\Gamma_i\backslash\mathsf{H})^{\mathrm{min}}$.)

Since $\mathrm{G} = \widetilde{\mathrm{G}}_0$ is connected, semisimple, and simply-connected, it factorizes as a product $\mathrm{G} \cong \prod_{j\in J} \mathrm{G}_j$ of its $\mathbb{Q}$-simple factors, which induces a factorization $\mathrm{M} \cong \prod_{j\in J} \mathrm{M}_j$. (We shall denote similar factorizations over J by subscripts $j \in J$, without explicitly introducing the other notations.) For each $j \in J$, let $\overline{\Gamma}_j$ denote the image of Γ under the canonical homomorphism $\mathrm{G} \to \mathrm{G}_j$, so that Γ is of finite index in $\overline{\Gamma} = \prod_{j\in J} \overline{\Gamma}_j$, and so that we have a finite morphism

$$\mathsf{X} = \Gamma\backslash\mathsf{H} \to \prod_{j\in J} \mathsf{X}_j \tag{3.3}$$

with $\mathsf{X}_j = \overline{\Gamma}_j\backslash\mathsf{H}_j$ for all $j \in J$, which extends to a finite morphism

$$\mathsf{X}^{\mathrm{min}} \to \prod_{j\in J} \mathsf{X}_j^{\mathrm{min}} \tag{3.4}$$

with $\mathsf{X}_j^{\mathrm{min}} = (\overline{\Gamma}_j\backslash\mathsf{H}_j)^{\mathrm{min}}$ for all $j \in J$. Up to replacing the cone decomposition for $\mathsf{X}^{\mathrm{tor}}$ with a further refinement (which we assume to be still projective and smooth), we may assume that (3.3) extends to a proper morphism

$$\mathsf{X}^{\mathrm{tor}} \to \prod_{j\in J} \mathsf{X}_j^{\mathrm{tor}} \tag{3.5}$$

with some noncanonical choices of toroidal compactifications $\mathsf{X}_j^{\mathrm{tor}} = (\Gamma_j\backslash\mathsf{H}_j)^{\mathrm{tor}}$ for all $j \in J$ (provided that the cone decomposition for $\mathsf{X}^{\mathrm{tor}}$ is finer than the pullback of the product cone decomposition for $\prod_{j\in J} \mathsf{X}_j^{\mathrm{tor}}$), which is compatible with (3.4).

For each $j \in J$, let $\nu_j \in \mathrm{X}^+_{\mathrm{M}_j}$ denote the factor of ν corresponding to the factor M_j of M. By assumption, there exist integers $N \geq 1$ and $N_j \geq 1$, for all $j \in J$, such that $N\nu_j = N_j(2\rho^{\mathrm{M}_j})$, and so that $\underline{W}^{\mathrm{can}}_{N\nu}$ over $\mathsf{X}^{\mathrm{tor}}$ is the pullback under (3.5) of $\boxtimes_j \underline{W}^{\mathrm{can}}_{N\nu_j} \cong \boxtimes_j (\Omega^{d_j}_{\mathsf{X}_j^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D}_j))^{\otimes N_j}$ over $\prod_{j\in J} \mathsf{X}_j^{\mathrm{tor}}$, where $d_j = \dim_{\mathbb{C}}(\mathsf{X}_j) = \dim_{\mathbb{C}}(\mathsf{X}_j^{\mathrm{tor}})$ and $\mathsf{D}_j = (\mathsf{X}_j^{\mathrm{tor}} - \mathsf{X}_j)_{\mathrm{red}}$ (with its reduced structure) for each $j \in J$. By [42, Prop. 3.4 b)], each $\Omega^{d_j}_{\mathsf{X}_j^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D}_j)$ over $\mathsf{X}_j^{\mathrm{tor}}$ is semiample and descends to an ample line bundle ω_j over $\mathsf{X}_j^{\mathrm{min}}$. Since (3.4) is finite, this shows that $\underline{W}^{\mathrm{can}}_{N\nu}$ is semiample and descends to an ample line bundle $\omega_{N\nu}$ over $\mathsf{X}^{\mathrm{min}}$, which is the pullback of the ample line bundle $\boxtimes_j \omega_j^{\otimes N_j}$ over $\prod_{j\in J} \mathsf{X}_j^{\mathrm{min}}$, as desired. □

Lemma 3.6 (cf. [35, property (5) preceding (2.1)] and [37, Prop. 4.2(5)])**.** *Under the assumption that* $\mathsf{X}^{\mathrm{tor}} \cong \coprod_i (\Gamma_i\backslash\mathsf{H}_0)^{\mathrm{tor}}_{\Sigma_i}$ *for some projective smooth cone decompositions* Σ_i*, there exists an effective Cartier divisor* D' *on* $\mathsf{X}^{\mathrm{tor}}$ *such that* $\mathsf{D}'_{\mathrm{red}} = \mathsf{D}$ *and such that* $\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}}(-\mathsf{D}')$ *is relatively ample over* $\mathsf{X}^{\mathrm{min}}$ *via the canonical proper surjective morphism* $\oint : \mathsf{X}^{\mathrm{tor}} \to \mathsf{X}^{\mathrm{min}}$*.*

Proof. By the results in [1, Ch. IV, Sec. 2], there exists some coherent $\mathscr{O}_{\mathsf{X}^{\min}}$-ideal $\mathscr{J}$ such that $\mathsf{X}^{\mathrm{tor}} \cong \mathrm{NBl}_{\mathscr{J}}(\mathsf{X}^{\min})$, the normalization of the blowup of $\mathsf{X}^{\min}$ at $\mathscr{J}$, and such that the pullback of $\mathscr{J}$ to $\mathsf{X}^{\mathrm{tor}}$ is a line bundle isomorphic to $\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}}(\mathsf{D}')$ for some effective Cartier divisor D' as in the statement of the lemma. □

Proposition 3.7 (cf. [35, (2.1)] and [37, (4.5)])**.** *There exists an effective Cartier divisor D' on $\mathsf{X}^{\mathrm{tor}}$ such that $\mathsf{D}'_{\mathrm{red}} = \mathsf{D}$, and such that, for any positive parallel weight $\nu \in \mathrm{X}^+_{\mathrm{M}}$ (see Definition 3.1), there exists some integer N_0 such that $\underline{W}^{\mathrm{can}}_{N\nu}(-\mathsf{D}')$ is ample for all $N \geq N_0$.*

Proof. Combine Lemmas 3.2 and 3.6. □

3.2. **Positive parallel weights of smallest sizes.**

Theorem 3.8. *For each $\alpha \in \Phi_{\mathrm{G}_{\mathbb{C}}}$, which necessarily comes from some $\mathbb{C}$-simple factor of $\widetilde{\mathrm{G}}_{0,\mathbb{C}}$, we have*

$$\left|\frac{(2\rho^{\mathrm{M}}, \alpha^\vee)}{h^\vee}\right| \in \begin{cases} \{0\}, & \text{if } \alpha \in \Phi_{\mathrm{M}} \text{ (i.e., compact as in Definition 2.1)}; \\ \{0,1\}, & \text{if the factor is not of types B or C}; \\ \{0,1,2\}, & \text{in all cases}; \end{cases} \tag{3.9}$$

where $h^\vee$ is the **dual Coxeter number** (*cf.* [26, Sec. 6.1]) *of the $\mathbb{C}$-simple factor of $\widetilde{\mathrm{G}}_{0,\mathbb{C}}$ from where $\alpha^\vee$ comes, which can be given explicitly as*

$$h^\vee = \begin{cases} n+1, & \text{if } \alpha^\vee \text{ comes from a } \mathbb{C}\text{-simple factor of type } \mathrm{A}_n; \\ 2n-1, & \text{if } \alpha^\vee \text{ comes from a } \mathbb{C}\text{-simple factor of type } \mathrm{B}_n; \\ n+1, & \text{if } \alpha^\vee \text{ comes from a } \mathbb{C}\text{-simple factor of type } \mathrm{C}_n; \\ 2n-2, & \text{if } \alpha^\vee \text{ comes from a } \mathbb{C}\text{-simple factor of type } \mathrm{D}_n; \\ 12, & \text{if } \alpha^\vee \text{ comes from a } \mathbb{C}\text{-simple factor of type } \mathrm{E}_6; \\ 18, & \text{if } \alpha^\vee \text{ comes from a } \mathbb{C}\text{-simple factor of type } \mathrm{E}_7. \end{cases} \tag{3.10}$$

Proof. Note that the assertion is only about the Lie algebras of $\mathrm{G}_{\mathbb{C}}$, P, and M (with some choices of B and T as above). Without loss of generality, we may and we shall replace $\mathrm{G}_{\mathbb{C}}$ with the $\mathbb{C}$-simple factors of $\widetilde{\mathrm{G}}_{0,\mathbb{C}}$, and assume that there is a unique simple $\alpha_0 \in \Phi^{\mathrm{M},+}_{\mathrm{G}_{\mathbb{C}}}$ (because the assertion to prove is trivial when $\alpha \in \Phi_{\mathrm{M}}$, by Lemma 2.2). By the classification of Hermitian symmetric domains (see, e.g., [23, Ch. X, Sec. 6, Table V]), we know that α_0 is a long root, and that $(\alpha, \alpha_0^\vee) = 3$ cannot happen for any $\alpha \in \Phi_{\mathrm{G}_{\mathbb{C}}}$. As explained in the proof of Lemma 2.3, $2\rho^{\mathrm{M}}$ is a positive multiple of the fundamental weight ϖ_0 dual to α_0, and it suffices to show that

$$(2\rho^{\mathrm{M}}, \alpha_0^\vee) = h^\vee, \tag{3.11}$$

because $\alpha_0^\vee$ appears in the expression of a noncompact coroot $\alpha^\vee$ with multiplicity at most two when $\mathrm{G}_{\mathbb{C}}$ is of types B or C, and at most one otherwise.

This can be easily checked in all cases by explicit calculations (cf. Section 3.3 below)—Indeed, this was how we observed the truth of this theorem. Nevertheless, we shall present a more conceptual argument, which we learned from Zhiwei Yun.

Let θ denote the highest root of $\mathrm{G}_{\mathbb{C}}$, and let $\theta^\vee$ denote the corresponding coroot. Essentially by definition, since $(\rho_{\mathrm{G}_{\mathbb{C}}}, \alpha^\vee) = 1$ for every positive simple root α, we have $h^\vee = 1 + (\rho_{\mathrm{G}_{\mathbb{C}}}, \theta^\vee)$. Since θ is the highest root, it is the only root $\alpha \in \Phi^+_{\mathrm{G}_{\mathbb{C}}}$ such that $(\alpha, \theta^\vee) = 2$. Since $(2\rho_{\mathrm{G}_{\mathbb{C}}}, \theta^\vee) = 2(h^\vee - 1)$, there are exactly $2(h^\vee - 2)$

(necessarily positive) roots $\alpha \in \Phi^+_{\mathrm{G}_\mathbb{C}}$ such that $(\alpha, \theta^\vee) = 1$. Since α_0 and θ are both long roots, they are in the same orbit of $\mathrm{W}_{\mathrm{G}_\mathbb{C}}$. Therefore, it is also true that there are exactly $2(h^\vee - 2)$ roots $\alpha \in \Phi_{\mathrm{G}_\mathbb{C}}$ such that $(\alpha, \alpha_0^\vee) = 1$.

Suppose $\alpha \in \Phi_{\mathrm{G}_\mathbb{C}}$ satisfies $(\alpha, \alpha_0^\vee) = 1$. Then $(\alpha + \alpha_0, \alpha_0^\vee) = 3$, which forces $\alpha + \alpha_0 \notin \Phi_{\mathrm{G}_\mathbb{C}}$. By [47, Lem. 9.1.3], it follows that $\alpha - \alpha_0 \in \Phi_{\mathrm{G}_\mathbb{C}}$, but $\alpha - 2\alpha_0 \notin \Phi_{\mathrm{G}_\mathbb{C}}$. Then we have two cases: (i) $\alpha \in \Phi_\mathrm{M}$ and $(\alpha, \alpha_0^\vee) = 1$; or (ii) $\alpha \in \Phi^{\mathrm{M},+}_{\mathrm{G}_\mathbb{C}}$, in which case we have $\beta = \alpha - \alpha_0 \in \Phi_\mathrm{M}$ satisfying $-\beta \in \Phi_\mathrm{M}$ and $(-\beta, \alpha_0^\vee) = 1$. Since the two cases have the same number of roots, there are $h^\vee - 2$ of them in each case. Thus, $(2\rho^\mathrm{M}, \alpha_0^\vee) = (\alpha_0, \alpha_0^\vee) + \sum\limits_{\alpha \text{ in case (ii)}} (\alpha, \alpha_0^\vee) = 2 + (h^\vee - 2) = h^\vee$, as desired. □

Remark 3.12. We learned from Xinwen Zhu that the assertion in Theorem 3.8 that $\frac{(2\rho^\mathrm{M}, \alpha^\vee)}{h^\vee}$ is an integer for all coroots $\alpha^\vee$ of $\mathrm{G}_\mathbb{C}$ is a special case of deeper investigations in [4, Sec. 4.6] and [52, Sec. 6.3] concerning Schubert subvarieties of affine Grassmannians. (The $\mathrm{G}^\circ_\mathbb{C}/\mathrm{P}$ considered here corresponds to Schubert subvarieties associated with minuscule cocharacters.)

Corollary 3.13. *Up to replacing* G *with* $\widetilde{\mathrm{G}}_0$ (*and replacing* M *etc with* $\widetilde{\mathrm{M}}_0$ *etc, accordingly*), *there exists a positive parallel weight* $\nu_+ \in \mathrm{X}^+_\mathrm{M}$ (*as in Definition* 3.1) *such that, for each coroot* $\alpha^\vee$ *of* $\mathrm{G}_\mathbb{C}$, *which necessarily comes from some* $\mathbb{C}$*-simple factor of* $\widetilde{\mathrm{G}}_{0,\mathbb{C}}$, *we have*

$$|(\nu_+, \alpha^\vee)| \leq \begin{cases} 0, & \text{if } \alpha \in \Phi_\mathrm{M} \text{ (i.e., compact as in Definition 2.1);} \\ 1, & \text{if the factor is not of types B or C;} \\ 2, & \text{in all cases.} \end{cases} \tag{3.14}$$

Such a ν_+ *is characterized by the property that its pullback to each* $\mathbb{C}$*-simple factor of* $\widetilde{\mathrm{G}}_{0,\mathbb{C}}$ *is the fundamental weight* ϖ_0 *dual to the unique simple* $\alpha_0 \in \Phi^{\mathrm{M},+}_{\mathrm{G}_\mathbb{C}}$ (*see Definition* 2.1) *from that* $\mathbb{C}$*-simple factor, when* α_0 *exists, or is zero otherwise.*

Proof. We may and we shall replace G with $\widetilde{\mathrm{G}}_0$ (and replace M etc with $\widetilde{\mathrm{M}}_0$ etc, accordingly), so that we have a factorization $\mathrm{G} \cong \prod\limits_{j \in J} \mathrm{G}_j$ into its $\mathbb{Q}$-simple factors, which induces a factorization $\mathrm{M} \cong \prod\limits_{j \in J} \mathrm{M}_j$, as in the proof of Lemma 3.2. Then we can write $\rho^\mathrm{M} = (\rho^{\mathrm{M}_j})_{j \in J}$, and it suffices to take $\nu_+ = (\frac{1}{h_j^\vee}(2\rho^{\mathrm{M}_j}))_{j \in J}$, where $h_j^\vee$ is the dual Coxeter number of any of the $\mathbb{C}$-simple factors of G_j, by Theorem 3.8 and its proof. (The upshot is that the multiple $\frac{1}{h_j^\vee}$ depends only on the $\mathbb{Q}$-simple factor G_j, but not on its further factorization into a product of $\mathbb{C}$-simple factors.) □

3.3. **Explicit descriptions in all cases.** For our main results to be stated in Section 4 to be practically useful, it is desirable to have explicit descriptions of positive parallel weights of $\mathrm{G}_\mathbb{C}$ in all cases. For this purpose, by Definition 3.1, it suffices to describe the pullback of such weights to the $\mathbb{Q}$-simple factors of $\widetilde{\mathrm{G}}_{0,\mathbb{C}}$. Hence, we may and we shall assume that $\mathrm{G}_\mathbb{C}$ is $\mathbb{Q}$-simple, and decomposes as a product $\mathrm{G}_\mathbb{C} \cong \prod\limits_{\upsilon \in \Upsilon} \mathrm{G}_\upsilon$ of its $\mathbb{C}$-simple factors, so that we have corresponding decompositions $\mathrm{P} \cong \prod\limits_{\upsilon \in \Upsilon} \mathrm{P}_\upsilon$, $\mathrm{M} \cong \prod\limits_{\upsilon \in \Upsilon} \mathrm{M}_\upsilon$, $\mathrm{X}_{\mathrm{G}_\mathbb{C}} = \prod\limits_{\upsilon \in \Upsilon} \mathrm{X}_{\mathrm{G}_\upsilon}$, $\mathrm{X}_\mathrm{M} = \prod\limits_{\upsilon \in \Upsilon} \mathrm{X}_{\mathrm{M}_\upsilon}$, $\Phi_\mathrm{G} = \coprod\limits_{\upsilon \in \Upsilon} \Phi_{\mathrm{G}_\upsilon}$, etc. Thanks to the classification of Hermitian symmetric domains (see, e.g., [23, Ch. X, Sec. 6, Table V]), we only have to investigate the following six cases. (Readers who are not interested can skip these and move on to the next section.)

3.3.1. *Type* A. Suppose that the root systems $\{\Phi_{\mathrm{G}_\upsilon}\}_{\upsilon\in\Upsilon}$ are all simple of type A_n for some integer n. For each $\upsilon \in \Upsilon$, let us embed $\Phi_{\mathrm{G}_\upsilon}$ into $(\mathbb{R}e)^\perp \subset \mathbb{R}^{n+1}$, where $e = (1, 1, \ldots, 1)$ has all its entries equal to 1, by taking the roots to be $e_i - e_j$, for $1 \leq i, j \leq n+1$ with $i \neq j$, where e_i and e_j are the i-th and j-th standard basis vectors of $\mathbb{R}^{n+1}$, with the Killing form induced by the standard inner product of $\mathbb{R}^{n+1}$. (By the r-th standard basis vector e_r, we mean the vector with the r-th entry being 1 and all other entries being 0.) For each root $\alpha = e_i - e_j$, the corresponding coroot is $\alpha^\vee = e_i - e_j$. Up to a change of coordinates, we shall assume that

$$\Phi^+_{\mathrm{G}_\upsilon} = \{e_i - e_j : 1 \leq i < j \leq n+1\}, \tag{3.15}$$

with positive simple roots given by $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n$, and that P_υ (when $\mathrm{M}_\upsilon \neq \mathrm{G}_\upsilon$) is determined by the condition that $\alpha_{r_\upsilon} \notin \Phi_{\mathrm{M}_\upsilon}$ for some $1 \leq r_\upsilon \leq n$. Then

$$\Phi^+_{\mathrm{M}_\upsilon} = \{e_i - e_j : 1 \leq i < j \leq r_\upsilon \text{ or } r_\upsilon < i < j \leq n+1\}, \tag{3.16}$$

whose elements are all perpendicular to the fundamental weight

$$\varpi_{r_\upsilon} = e_1 + \cdots + e_{r_\upsilon} = -(e_{r_\upsilon+1} + \cdots + e_{n+1}) \pmod{\mathbb{Z}e}, \tag{3.17}$$

while

$$\Phi^{\mathrm{M}_\upsilon,+}_{\mathrm{G}_\upsilon} = \{e_i - e_j : 1 \leq i \leq r_\upsilon < j \leq n+1\}. \tag{3.18}$$

Note that $\#\Phi^+_{\mathrm{G}_\upsilon} = \frac{1}{2}n(n+1)$, $\#\Phi^+_{\mathrm{M}_\upsilon} = \frac{1}{2}(r_\upsilon - 1)r_\upsilon + \frac{1}{2}(n - r_\upsilon)(n - r_\upsilon + 1)$, and $\#\Phi^{\mathrm{M}_\upsilon,+}_{\mathrm{G}_\upsilon} = r_\upsilon(n - r_\upsilon + 1)$, where the first one is the sum of the latter two. Hence,

$$\rho_{\mathrm{G}_\upsilon} = \tfrac{1}{2}(n, n-2, \ldots, 2-n, -n), \tag{3.19}$$

$$\rho_{\mathrm{M}_\upsilon} = \tfrac{1}{2}(r_\upsilon - 1, r_\upsilon - 3, \ldots, 1 - r_\upsilon; n - r_\upsilon, n - r_\upsilon - 2, \ldots, r_\upsilon - n), \tag{3.20}$$

and

$$\begin{aligned}\rho^{\mathrm{M}_\upsilon} &= \rho_{\mathrm{G}_\upsilon} - \rho_{\mathrm{M}_\upsilon} \\ &= \tfrac{1}{2}(n - r_\upsilon + 1, n - r_\upsilon + 1, \ldots, n - r_\upsilon + 1; -r_\upsilon, -r_\upsilon, \ldots, -r_\upsilon) \\ &= \tfrac{1}{2}(n+1, n+1, \ldots, n+1; 0, 0, \ldots, 0) \pmod{\mathbb{Z}e} \\ &= \tfrac{1}{2}(0, 0, \ldots, 0; -n-1, -n-1, \ldots, -n-1) \pmod{\mathbb{Z}e} \\ &= \tfrac{n+1}{2}\varpi_{r_\upsilon} \pmod{\mathbb{Z}e},\end{aligned} \tag{3.21}$$

where the semicolons are after the r_υ-th entries. Since the highest root is

$$\theta = e_1 - e_{n+1} = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \tag{3.22}$$

so that $\theta^\vee = e_1 - e_{n+1}$ as well, we have

$$h^\vee = 1 + (\rho_{\mathrm{G}_\upsilon}, \theta^\vee) = n + 1. \tag{3.23}$$

Consequently, for each coroot $\alpha^\vee$ of G_υ, we have

$$\left|\frac{(2\rho^{\mathrm{M}_\upsilon}, \alpha^\vee)}{h^\vee}\right| = \begin{cases} 1, & \text{if } \alpha^\vee = \pm(e_i - e_j) \text{ with } 1 \leq i \leq r_\upsilon < j \leq n+1; \\ 0, & \text{otherwise.} \end{cases} \tag{3.24}$$

(In particular, we have reconfirmed Theorem 3.8 for all simple factors of type A.)

Lemma 3.25. *In this case, $\nu = (\nu_\upsilon)_{\upsilon\in\Upsilon} \in \mathrm{X}^+_{\mathrm{M}}$ is positive parallel if and only if there exists $k \in \mathbb{Z}_{\geq 1}$ such that, for each $\upsilon \in \Upsilon$, either $\mathrm{M}_\upsilon = \mathrm{G}_\upsilon$ and $\nu_\upsilon = 0$, or $\mathrm{M}_\upsilon \neq \mathrm{G}_\upsilon$ and*

$$\nu_\upsilon = k\varpi_{r_\upsilon} = (k, k, \ldots, k; 0, 0, \ldots, 0) \pmod{\mathbb{Z}e} \tag{3.26}$$

(*where the semicolon is after the r_υ-th entry*).

3.3.2. *Type* B. Suppose that the root systems $\{\Phi_{\mathrm{G}_\upsilon}\}_{\upsilon\in\Upsilon}$ are all simple of type B_n for some integer n. For each $\upsilon \in \Upsilon$, let us embed $\Phi_{\mathrm{G}_\upsilon}$ in $\mathbb{R}^n$ by taking the roots to be $\pm e_i \pm e_j$ (allowing all four possibilities of signs) and $\pm e_i$ for $1 \leq i, j \leq n$ with $i \neq j$, where e_i and e_j are i-th and j-th standard basis vectors of $\mathbb{R}^n$, with the Killing form induced by the standard inner product of $\mathbb{R}^n$. For each root $\alpha = \pm e_i \pm e_j$ (resp. $\pm e_i$), the corresponding coroot is $\alpha^\vee = \pm e_i \pm e_j$ (resp. $\pm 2e_i$). Up to a change of coordinates, we shall assume that

$$\Phi^+_{\mathrm{G}_\upsilon} = \{e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{e_i : 1 \leq i \leq n\}, \tag{3.27}$$

with positive simple roots given by $\alpha_i = e_i - e_{i+1}$ for $1 \leq i < n$ and $\alpha_n = e_n$, and that P_υ (when $\mathrm{M}_\upsilon \neq \mathrm{G}_\upsilon$) is determined by the condition that $\alpha_1 \notin \Phi_{\mathrm{M}_\upsilon}$. Then

$$\Phi^+_{\mathrm{M}_\upsilon} = \{e_i \pm e_j : 1 < i < j \leq n\} \cup \{e_i : 1 < i \leq n\}, \tag{3.28}$$

whose elements are all perpendicular to the fundamental weight

$$\varpi_1 = e_1 = (1, 0, 0, \ldots, 0), \tag{3.29}$$

while

$$\Phi^{\mathrm{M}_\upsilon,+}_{\mathrm{G}_\upsilon} = \{e_1 \pm e_j : 1 < j \leq n\} \cup \{e_1\} \tag{3.30}$$

Note that $\#\Phi^+_{\mathrm{G}_\upsilon} = n^2$, $\#\Phi^+_{\mathrm{M}_\upsilon} = (n-1)^2$, and $\#\Phi^{\mathrm{M}_\upsilon,+}_{\mathrm{G}_\upsilon} = 2n-1$, where the first one is the sum of the latter two. Hence,

$$\rho_{\mathrm{G}_\upsilon} = \tfrac{1}{2}(2n-1, 2n-3, \ldots, 3, 1), \tag{3.31}$$

$$\rho_{\mathrm{M}_\upsilon} = \tfrac{1}{2}(0; 2n-3, \ldots, 3, 1), \tag{3.32}$$

and

$$\rho^{\mathrm{M}_\upsilon} = \rho_{\mathrm{G}_\upsilon} - \rho_{\mathrm{M}_\upsilon} = \tfrac{1}{2}(2n-1; 0, 0, \ldots, 0) = \tfrac{2n-1}{2}\varpi_1. \tag{3.33}$$

Since the highest root is

$$\theta = e_1 + e_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n, \tag{3.34}$$

so that $\theta^\vee = e_1 + e_2$ as well, we have

$$h^\vee = 1 + (\rho_{\mathrm{G}_\upsilon}, \theta^\vee) = 2n - 1. \tag{3.35}$$

Consequently, for each coroot $\alpha^\vee$ of G_υ, we have

$$\left|\frac{(2\rho^{\mathrm{M}_\upsilon}, \alpha^\vee)}{h^\vee}\right| = \begin{cases} 2, & \text{if } \alpha^\vee = \pm 2e_1; \\ 1, & \text{if } \alpha^\vee = \pm e_1 \pm e_j \text{ with } 1 < j \leq n; \\ 0, & \text{otherwise.} \end{cases} \tag{3.36}$$

(In particular, we have reconfirmed Theorem 3.8 for all simple factors of type B.)

Lemma 3.37. *In this case, $\nu = (\nu_\upsilon)_{\upsilon\in\Upsilon} \in \mathrm{X}^+_{\mathrm{M}}$ is positive parallel if and only if there exists $k \in \mathbb{Z}_{\geq 1}$ such that, for each $\upsilon \in \Upsilon$, either $\mathrm{M}_\upsilon = \mathrm{G}_\upsilon$ and $\nu_\upsilon = 0$, or $\mathrm{M}_\upsilon \neq \mathrm{G}_\upsilon$ and*

$$\nu_\upsilon = k\varpi_1 = (k; 0, 0, \ldots, 0). \tag{3.38}$$

3.3.3. *Type* C. Suppose that the root systems $\{\Phi_{\mathrm{G}_\upsilon}\}_{\upsilon\in\Upsilon}$ are all simple of type C_n for some integer n. For each $\upsilon \in \Upsilon$, let us embed $\Phi_{\mathrm{G}_\upsilon}$ in $\mathbb{R}^n$ by taking the roots to be $\pm e_i \pm e_j$ (allowing all four possibilities of signs) and $\pm 2e_i$ for $1 \leq i, j \leq n$ with $i \neq j$, where e_i and e_j are i-th and j-th standard basis vectors of $\mathbb{R}^n$, with the Killing form induced by the standard inner product of $\mathbb{R}^n$. For each root $\alpha = \pm e_i \pm e_j$ (resp. $\pm 2e_i$), the corresponding coroot is $\alpha^\vee = \pm e_i \pm e_j$ (resp. $\pm e_i$). Up to a change of coordinates, we shall assume that

$$\Phi^+_{\mathrm{G}_\upsilon} = \{e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{2e_i : 1 \leq i \leq n\}, \tag{3.39}$$

with positive simple roots given by $\alpha_i = e_i - e_{i+1}$ for $1 \leq i < n$ and $\alpha_n = 2e_n$, and that P_υ (when $\mathrm{M}_\upsilon \neq \mathrm{G}_\upsilon$) is determined by the condition that $\alpha_n \notin \Phi_{\mathrm{M}_\upsilon}$. Then

$$\Phi^+_{\mathrm{M}_\upsilon} = \{e_i - e_j : 1 \leq i < j \leq n\}, \tag{3.40}$$

whose elements are all perpendicular to the fundamental weight

$$\varpi_n = e_1 + e_2 + \cdots + e_n = (1, 1, \ldots, 1), \tag{3.41}$$

while the positive noncompact roots are

$$\Phi^{\mathrm{M}_\upsilon,+}_{\mathrm{G}_\upsilon} = \{e_i + e_j : 1 \leq i < j \leq n\} \cup \{2e_i : 1 \leq i \leq n\} \tag{3.42}$$

Note that $\#\Phi^+_{\mathrm{G}_\upsilon} = n^2$, $\#\Phi^+_{\mathrm{M}_\upsilon} = \frac{1}{2}n(n-1)$, and $\#\Phi^{\mathrm{M}_\upsilon,+}_{\mathrm{G}_\upsilon} = \frac{1}{2}n(n+1)$, where the first one is the sum of the latter two. Hence,

$$\rho_{\mathrm{G}_\upsilon} = (n, n-1, \ldots, 2, 1), \tag{3.43}$$

$$\rho_{\mathrm{M}_\upsilon} = \tfrac{1}{2}(n-1, n-3, \ldots, 1-n), \tag{3.44}$$

and

$$\rho^{\mathrm{M}_\upsilon} = \rho_{\mathrm{G}_\upsilon} - \rho_{\mathrm{M}_\upsilon} = \tfrac{1}{2}(n+1, n+1, \ldots, n+1) = \tfrac{n+1}{2}\varpi_n. \tag{3.45}$$

Since the highest root is

$$\theta = 2e_1 = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n, \tag{3.46}$$

so that $\theta^\vee = e_1$, we have

$$h^\vee = 1 + (\rho_{\mathrm{G}_\upsilon}, \theta^\vee) = n + 1 \tag{3.47}$$

Consequently, for each coroot $\alpha^\vee$ of G_υ, we have

$$\left|\frac{(2\rho^{\mathrm{M}_\upsilon}, \alpha^\vee)}{h^\vee}\right| = \begin{cases} 2, & \text{if } \alpha^\vee = \pm(e_i + e_i) \text{ with } 1 \leq i < j \leq n; \\ 1, & \text{if } \alpha^\vee = \pm e_i \text{ with } 1 \leq i \leq n; \\ 0, & \text{otherwise.} \end{cases} \tag{3.48}$$

(In particular, we have reconfirmed Theorem 3.8 for all simple factors of type C.)

Lemma 3.49. *In this case, $\nu = (\nu_\upsilon)_{\upsilon\in\Upsilon} \in \mathrm{X}^+_{\mathrm{M}}$ is positive parallel if and only if there exists $k \in \mathbb{Z}_{\geq 1}$ such that, for each $\upsilon \in \Upsilon$, either $\mathrm{M}_\upsilon = \mathrm{G}_\upsilon$ and $\nu_\upsilon = 0$, or $\mathrm{M}_\upsilon \neq \mathrm{G}_\upsilon$ and*

$$\nu_\upsilon = k\varpi_n = (k, k, k, \ldots, k). \tag{3.50}$$

3.3.4. *Type* D. Suppose that the root systems $\{\Phi_{\mathrm{G}_\upsilon}\}_{\upsilon\in\Upsilon}$ are all simple of type D_n for some integer $n \geq 4$. (The case with $n = 3$ can be considered as the case A_3.) For each $\upsilon \in \Upsilon$, let us embed $\Phi_{\mathrm{G}_\upsilon}$ in $\mathbb{R}^n$ by taking the roots to be $\pm e_i \pm e_j$ (allowing all four possibilities of signs) for $1 \leq i, j \leq n$ with $i \neq j$, where e_i and e_j are i-th and j-th standard basis vectors of $\mathbb{R}^n$, with the Killing form induced by the standard inner product of $\mathbb{R}^n$. For each root α as above, the corresponding coroot $\alpha^\vee$ is exactly the same vector in $\mathbb{R}^n$. Up to a change of coordinates, we shall assume that

$$\Phi^+_{\mathrm{G}_\upsilon} = \{e_i \pm e_j : 1 \leq i < j \leq n\}, \tag{3.51}$$

with positive simple roots given by $\alpha_i = e_i - e_{i+1}$ for $1 \leq i < n$ and $\alpha_n = e_{n-1} + e_n$, and that P_υ (when $\mathrm{M}_\upsilon \neq \mathrm{G}_\upsilon$) is determined by the condition that $\alpha_{r_\upsilon} \notin \Phi_{\mathrm{M}_\upsilon}$ for exactly one index r_υ in $\{1, n-1, n\}$. The two cases $r_\upsilon = n-1$ and $r_\upsilon = n$ are essentially the same, up to a change of sign in the n-th coordinate. Hence, for simplicity, we shall omit the case $\alpha_{n-1} \notin \Phi_{\mathrm{M}_\upsilon}$.

Suppose $\alpha_1 \notin \Phi_{\mathrm{M}_\upsilon}$. (We shall say that we are in the case of type $\mathrm{D}^{\mathbb{R}}_n$.) Then

$$\Phi^+_{\mathrm{M}_\upsilon} = \{e_i \pm e_j : 1 < i < j \leq n\}, \tag{3.52}$$

which are all perpendicular to the fundamental weight

$$\varpi_1 = e_1 = (1, 0, 0, \ldots, 0), \tag{3.53}$$

while

$$\Phi^{\mathrm{M}_\upsilon,+}_{\mathrm{G}_\upsilon} = \{e_1 \pm e_j : 1 < j \leq n\} \tag{3.54}$$

Note that $\#\Phi^+_{\mathrm{G}_\upsilon} = n(n-1)$, $\#\Phi^+_{\mathrm{M}_\upsilon} = (n-1)(n-2)$, and $\#\Phi^{\mathrm{M}_\upsilon,+}_{\mathrm{G}_\upsilon} = 2n-2$, where the first one is the sum of the latter two. Hence,

$$\rho_{\mathrm{G}_\upsilon} = (n-1, n-2, \ldots, 1, 0), \tag{3.55}$$

$$\rho_{\mathrm{M}_\upsilon} = (0; n-2, n-3, \ldots, 1, 0), \tag{3.56}$$

and

$$\rho^{\mathrm{M}_\upsilon} = \rho_{\mathrm{G}_\upsilon} - \rho_{\mathrm{M}_\upsilon} = (n-1; 0, 0, \ldots, 0) = (n-1)\varpi_1. \tag{3.57}$$

Since the highest root is

$$\theta = e_1 + e_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \tag{3.58}$$

so that $\theta^\vee = e_1 + e_2$ as well, we have

$$h^\vee = 1 + (\rho_{\mathrm{G}_\upsilon}, \theta^\vee) = 2n-2. \tag{3.59}$$

Consequently, for each coroot $\alpha^\vee$ of G_υ, we have

$$\left|\frac{(2\rho^{\mathrm{M}_\upsilon}, \alpha^\vee)}{h^\vee}\right| = \begin{cases} 1, & \text{if } \alpha^\vee = \pm e_1 \pm e_j \text{ with } 1 < j \leq n; \\ 0, & \text{otherwise.} \end{cases} \tag{3.60}$$

Suppose $\alpha_n \notin \Phi_{\mathrm{M}_\upsilon}$. (We shall say that we are in the case of type $\mathrm{D}^{\mathbb{H}}_n$.) Then

$$\Phi^+_{\mathrm{M}_\upsilon} = \{e_i - e_j : 1 \leq i < j \leq n\}, \tag{3.61}$$

whose elements are all perpendicular to the fundamental weight

$$\varpi_n = \tfrac{1}{2}(e_1 + e_2 + \cdots + e_n) = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}), \tag{3.62}$$

while

$$\Phi^{\mathrm{M}_\upsilon,+}_{\mathrm{G}_\upsilon} = \{e_i + e_j : 1 \leq i < j \leq n\} \tag{3.63}$$

Note that $\#\Phi^+_{\mathrm{G}_\upsilon} = n(n-1)$, $\#\Phi^+_{\mathrm{M}_\upsilon} = \frac{1}{2}n(n-1)$, and $\#\Phi^{\mathrm{M}_\upsilon,+}_{\mathrm{G}_\upsilon} = \frac{1}{2}n(n-1)$, where the first one is the sum of the latter two. Hence,

$$\rho_{\mathrm{G}_\upsilon} = (n-1, n-2, \ldots, 1, 0), \tag{3.64}$$

$$\rho_{\mathrm{M}_\upsilon} = \tfrac{1}{2}(n-1, n-3, \ldots, 1-n), \tag{3.65}$$

and

$$\rho^{\mathrm{M}_\upsilon} = \rho_{\mathrm{G}_\upsilon} - \rho_{\mathrm{M}_\upsilon} = \tfrac{1}{2}(n-1, n-1, \ldots, n-1) = \tfrac{2n-2}{2}\varpi_n. \tag{3.66}$$

Consequently, for each coroot $\alpha^\vee$ of G_υ, we have

$$\left|\frac{(2\rho^{\mathrm{M}_\upsilon}, \alpha^\vee)}{h^\vee}\right| = \begin{cases} 1, & \text{if } \alpha^\vee = \pm(e_i + e_j) \text{ with } 1 \leq i < j \leq n; \\ 0, & \text{otherwise.} \end{cases} \tag{3.67}$$

(In particular, we have reconfirmed Theorem 3.8 for all simple factors of type D.)

Lemma 3.68. *In this case, $\nu = (\nu_\upsilon)_{\upsilon\in\Upsilon} \in \mathrm{X}^+_{\mathrm{M}}$ is positive parallel if and only if there exists $k \in \mathbb{Z}_{\geq 1}$ such that, for each $\upsilon \in \Upsilon$, either $\mathrm{M}_\upsilon = \mathrm{G}_\upsilon$ and $\nu_\upsilon = 0$, or $\mathrm{M}_\upsilon \neq \mathrm{G}_\upsilon$ and*

$$\nu_\upsilon = \begin{cases} k\varpi_1 = (k; 0, 0, \ldots, 0), & \text{if } \alpha_1 = e_1 - e_2 \notin \Phi_{\mathrm{M}_\upsilon}; \\ k\varpi_{n-1} = (\frac{k}{2}, \frac{k}{2}, \ldots, \frac{k}{2}, -\frac{k}{2}), & \text{if } \alpha_{n-1} = e_{n-1} - e_n \notin \Phi_{\mathrm{M}_\upsilon}; \\ k\varpi_n = (\frac{k}{2}, \frac{k}{2}, \ldots, \frac{k}{2}, \frac{k}{2}), & \text{if } \alpha_n = e_{n-1} + e_n \notin \Phi_{\mathrm{M}_\upsilon}. \end{cases} \tag{3.69}$$

3.3.5. *Type* E_6. Suppose that the root systems $\{\Phi_{\mathrm{G}_\upsilon}\}_{\upsilon\in\Upsilon}$ are all simple of type E_6. For each $\upsilon \in \Upsilon$, let us embed $\Phi_{\mathrm{G}_\upsilon}$ in $\mathbb{R}^6$ by taking the 72 roots to be all 40 possibilities of $\pm e_i \pm e_j$ (allowing all four possibilities of signs) with $1 \leq i < j \leq 5$, where e_i and e_j are i-th and j-th standard basis vectors of $\mathbb{R}^6$ as usual, together with all 32 possibilities of $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ with an odd number of positive signs, with the Killing form induced by the standard inner product of $\mathbb{R}^6$. For each root α as above, the corresponding coroot $\alpha^\vee$ is exactly the same vector in $\mathbb{R}^6$. Up to a change of coordinates, we shall assume that

$$\begin{aligned} \Phi^+_{\mathrm{G}_\upsilon} = &\{e_i \pm e_j : 1 \leq i < j \leq 5\} \\ &\cup \{(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, +\tfrac{\sqrt{3}}{2}) \text{ with an odd number of } +\text{'s }\}, \end{aligned} \tag{3.70}$$

with positive simple roots given by $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3 - e_4$, $\alpha_4 = e_4 - e_5$, $\alpha_5 = e_4 + e_5$, and $\alpha_6 = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2})$, and that P_υ (when $\mathrm{M}_\upsilon \neq \mathrm{G}_\upsilon$) is determined by the condition that $\alpha_{r_\upsilon} \notin \Phi_{\mathrm{M}_\upsilon}$ for exactly one index r_υ in $\{1, 6\}$. While the two cases are essentially the same, they are quite different for explicit calculations. Hence, we shall still treat them separately.

Suppose $\alpha_1 \notin \Phi_{\mathrm{M}_\upsilon}$. Then

$$\begin{aligned} \Phi^+_{\mathrm{M}_\upsilon} = &\{e_i \pm e_j : 1 < i < j \leq 5\} \\ &\cup \{(-\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, +\tfrac{\sqrt{3}}{2}) \text{ with an odd number of } +\text{'s }\} \end{aligned} \tag{3.71}$$

whose elements are all perpendicular to the fundamental weight

$$\varpi_1 = (1, 0, 0, 0, 0, \tfrac{\sqrt{3}}{3}), \tag{3.72}$$

while

$$\begin{aligned} \Phi^{\mathrm{M}_\upsilon,+}_{\mathrm{G}_\upsilon} = &\{e_1 \pm e_j : 1 < j \leq 5\} \\ &\cup \{(+\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, +\tfrac{\sqrt{3}}{2}) \text{ with an odd number of } +\text{'s }\}. \end{aligned} \tag{3.73}$$

Note that $\#\Phi^+_{\mathrm{G}_v} = 36$, $\#\Phi^+_{\mathrm{M}_v} = 12 + 8 = 20$, and $\#\Phi^{\mathrm{M}_v,+}_{\mathrm{G}_v} = 8 + 8 = 16$, where the first one is the sum of the latter two. Hence,

$$\rho_{\mathrm{G}_v} = (4, 3, 2, 1, 0, 4\sqrt{3}), \tag{3.74}$$

$$\rho_{\mathrm{M}_v} = (-2; 3, 2, 1, 0, 2\sqrt{3}), \tag{3.75}$$

and

$$\rho^{\mathrm{M}_v} = \rho_{\mathrm{G}_v} - \rho_{\mathrm{M}_v} = (6; 0, 0, 0, 0, 2\sqrt{3}) = 6\varpi_1. \tag{3.76}$$

Since the highest root is

$$\theta = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6, \tag{3.77}$$

so that $\theta^\vee = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2})$ as well, we have

$$h^\vee = 1 + (\rho_{\mathrm{G}_v}, \theta^\vee) = 12. \tag{3.78}$$

Consequently, for each coroot $\alpha^\vee$ of G_v, we have

$$\left|\frac{(2\rho^{\mathrm{M}_v}, \alpha^\vee)}{h^\vee}\right| = \begin{cases} 1, & \text{if } \alpha^\vee = \pm e_1 \pm e_j \text{ with } 1 < j \leq 5; \\ 1, & \text{if } \alpha^\vee = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) \\ & \text{with an odd number of +'s and} \\ & \text{with the first sign equal to the last sign;} \\ 0, & \text{otherwise.} \end{cases} \tag{3.79}$$

Suppose $\alpha_6 \notin \Phi_{\mathrm{M}_v}$. Then

$$\Phi^+_{\mathrm{M}_v} = \{e_i \pm e_j : 1 \leq i < j \leq 5\} \tag{3.80}$$

which are all perpendicular to the fundamental weight

$$\varpi_6 = (0, 0, 0, 0, 0, \tfrac{2\sqrt{3}}{3}), \tag{3.81}$$

while

$$\Phi^{\mathrm{M}_v,+}_{\mathrm{G}_v} = \{(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, +\tfrac{\sqrt{3}}{2}) \text{ with an odd number of +'s }\}. \tag{3.82}$$

Note that $\#\Phi^+_{\mathrm{G}_v} = 36$, $\#\Phi^+_{\mathrm{M}_v} = 20$, and $\#\Phi^{\mathrm{M}_v,+}_{\mathrm{G}_v} = 16$, where the first one is the sum of the latter two. Hence,

$$\rho_{\mathrm{G}_v} = (4, 3, 2, 1, 0, 4\sqrt{3}), \tag{3.83}$$

$$\rho_{\mathrm{M}_v} = (4, 3, 2, 1, 0; 0), \tag{3.84}$$

and

$$\rho^{\mathrm{M}_v} = \rho_{\mathrm{G}_v} - \rho_{\mathrm{M}_v} = (0, 0, 0, 0, 0; 4\sqrt{3}) = 6\varpi_6. \tag{3.85}$$

Consequently, for each coroot $\alpha^\vee$ of G_v, we have

$$\left|\frac{(2\rho^{\mathrm{M}_v}, \alpha^\vee)}{h^\vee}\right| = \begin{cases} 1, & \text{if } \alpha^\vee = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) \\ & \text{with an odd number of +'s;} \\ 0, & \text{otherwise.} \end{cases} \tag{3.86}$$

(In particular, we have reconfirmed Theorem 3.8 for all simple factors of type E_6.)

Lemma 3.87. *In this case,* $\nu = (\nu_\upsilon)_{\upsilon\in\Upsilon} \in \mathrm{X}^+_{\mathrm{M}}$ *is positive parallel if and only if there exists* $k \in \mathbb{Z}_{\geq 1}$ *such that, for each* $\upsilon \in \Upsilon$*, either* $\mathrm{M}_\upsilon = \mathrm{G}_\upsilon$ *and* $\nu_\upsilon = 0$*, or* $\mathrm{M}_\upsilon \neq \mathrm{G}_\upsilon$ *and*

(3.88)
$$\nu_\upsilon = \begin{cases} k\varpi_1 = (k, 0, 0, 0, 0, \frac{\sqrt{3}}{3}k), & \text{if } \alpha_1 = e_1 - e_2 \notin \Phi_{\mathrm{M}_\upsilon}; \\ k\varpi_6 = (0, 0, 0, 0, 0, \frac{2\sqrt{3}}{3}k), & \text{if } \alpha_6 = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}) \notin \Phi_{\mathrm{M}_\upsilon}. \end{cases}$$

3.3.6. *Type* E_7. Suppose that the root systems $\{\Phi_{\mathrm{G}_\upsilon}\}_{\upsilon\in\Upsilon}$ are all simple of type E_7. For each $\upsilon \in \Upsilon$, let us embed $\Phi_{\mathrm{G}_\upsilon}$ in $\mathbb{R}^7$ by taking the 126 roots to be all 60 possibilities of $\pm e_i \pm e_j$ (allowing all four possibilities of signs) with $1 \leq i < j \leq 6$, where e_i and e_j are i-th and j-th standard basis vectors of $\mathbb{R}^7$ as usual, together with all 64 possibilities of $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2})$ with an even number of $+\frac{1}{2}$'s and the 2 possibilities of $(0, 0, 0, 0, 0, 0, \pm\sqrt{2})$, with the Killing form induced by the standard inner product of $\mathbb{R}^7$. For each root α as above, the corresponding coroot $\alpha^\vee$ is exactly the same vector in $\mathbb{R}^7$. Up to a change of coordinates, we shall assume that

$$\begin{aligned} \Phi^+_{\mathrm{G}_\upsilon} = & \{e_i \pm e_j : 1 \leq i < j \leq 6\} \\ & \cup \{(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \tfrac{\sqrt{2}}{2}) \text{ with an even number of } +\tfrac{1}{2}\text{'s}\} \\ & \cup \{(0, 0, 0, 0, 0, 0, \sqrt{2})\}, \end{aligned} \tag{3.89}$$

with positive simple roots given by $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3 - e_4$, $\alpha_4 = e_4 - e_5$, $\alpha_5 = e_5 - e_6$, $\alpha_6 = e_5 + e_6$, and $\alpha_7 = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2})$, and that P_υ (when $\mathrm{M}_\upsilon \neq \mathrm{G}_\upsilon$) is determined by the condition that $\alpha_1 \notin \Phi_{\mathrm{M}_\upsilon}$. Then

$$\begin{aligned} \Phi^+_{\mathrm{M}_\upsilon} = & \{e_i \pm e_j : 1 < i < j \leq 6\} \\ & \cup \{(-\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \tfrac{\sqrt{2}}{2}) \text{ with an even number of } +\tfrac{1}{2}\text{'s}\} \end{aligned} \tag{3.90}$$

whose elements are all perpendicular to the fundamental weight

$$\varpi_1 = (1, 0, 0, 0, 0, 0, \tfrac{\sqrt{2}}{2}), \tag{3.91}$$

while

$$\begin{aligned} \Phi^{\mathrm{M}_\upsilon,+}_{\mathrm{G}_\upsilon} = & \{e_1 \pm e_j : 1 < j \leq 6\} \\ & \cup \{(\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \tfrac{\sqrt{2}}{2}) \text{ with an even number of } +\tfrac{1}{2}\text{'s}\} \\ & \cup \{(0, 0, 0, 0, 0, 0, \sqrt{2})\}. \end{aligned} \tag{3.92}$$

Note that $\#\Phi^+_{\mathrm{G}_\upsilon} = 63$, $\#\Phi^+_{\mathrm{M}_\upsilon} = 20 + 16 = 36$, and $\#\Phi^{\mathrm{M}_\upsilon,+}_{\mathrm{G}_\upsilon} = 10 + 16 + 1 = 27$, where the first one is the sum of the latter two. Hence,

$$\rho_{\mathrm{G}_\upsilon} = (5, 4, 3, 2, 1, 0, \tfrac{17\sqrt{2}}{2}), \tag{3.93}$$

$$\rho_{\mathrm{M}_\upsilon} = (-4; 4, 3, 2, 1, 0, 4\sqrt{2}), \tag{3.94}$$

and

$$\rho^{\mathrm{M}_\upsilon} = \rho_{\mathrm{G}_\upsilon} - \rho_{\mathrm{M}_\upsilon} = (9; 0, 0, 0, 0, 0, \tfrac{9\sqrt{2}}{2}) = 9\varpi_1. \tag{3.95}$$

Since the highest root is

$$\theta = (0, 0, 0, 0, 0, 0, \sqrt{2}) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6 + 2\alpha_7, \tag{3.96}$$

so that $\theta^\vee = (0,0,0,0,0,0,\sqrt{2})$ as well, we have

$$h^\vee = 1 + (\rho_{\mathrm{G}_\upsilon}, \theta^\vee) = 18. \tag{3.97}$$

Consequently, for each coroot $\alpha^\vee$ of G_υ, we have

$$\left| \frac{(2\rho^{\mathrm{M}_\upsilon}, \alpha^\vee)}{h^\vee} \right| = \begin{cases} 1, & \text{if } \alpha^\vee = \pm e_1 \pm e_j \text{ with } 1 < j \leq 6; \\ 1, & \text{if } \alpha^\vee = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}) \\ & \text{with an even number of } +\frac{1}{2}\text{'s and} \\ & \text{with the first sign equal to the last sign;} \\ 0, & \text{otherwise.} \end{cases} \tag{3.98}$$

(In particular, we have reconfirmed Theorem 3.8 for all simple factors of type E_7.)

Lemma 3.99. *In this case, $\nu = (\nu_\upsilon)_{\upsilon \in \Upsilon} \in \mathrm{X}^+_{\mathrm{M}}$ is positive parallel if and only if there exists $k \in \mathbb{Z}_{\geq 1}$ such that, for each $\upsilon \in \Upsilon$, either $\mathrm{M}_\upsilon = \mathrm{G}_\upsilon$ and $\nu_\upsilon = 0$, or $\mathrm{M}_\upsilon \neq \mathrm{G}_\upsilon$ and*

$$\nu_\upsilon = k\varpi_1 = (k,0,0,0,0,0,\tfrac{\sqrt{2}}{2}k). \tag{3.100}$$

4. Main results

4.1. Vanishing of coherent cohomology.

Let $d := \dim_{\mathbb{C}}(\mathsf{X}) = \dim_{\mathbb{C}}(\mathsf{H})$.

Theorem 4.1 (cf. [36, Thm. 8.7 and 8.20] and [37, Thm. 8.13 and 8.23])**.** *Let $\nu \in \mathrm{X}^+_{\mathrm{M}}$. With the terminologies in Definitions 2.6 and 3.1, we have:*

(1) If there exists a positive parallel weight ν_- such that $\nu+\nu_-$ is cohomological, then $H^i(\mathsf{X}^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_\nu) = 0$ for every $i < d - l(w(\nu+\nu_-))$.

(2) If there exists a positive parallel weight ν_+ such that $\nu-\nu_+$ is cohomological, then $H^i(\mathsf{X}^{\mathrm{tor}}, \underline{W}^{\mathrm{sub}}_\nu) = 0$ for every $i > d - l(w(\nu-\nu_+))$.

(3) If there exist positive parallel weights ν_+ and ν_- such that $\nu-\nu_+$ and $\nu+\nu_-$ are both cohomological, then the interior cohomology

$$H^i_{\mathrm{int}}(\mathsf{X}^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_\nu) := \mathrm{image}(H^i(\mathsf{X}^{\mathrm{tor}}, \underline{W}^{\mathrm{sub}}_\nu) \to H^i(\mathsf{X}^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_\nu)) = 0$$

for every $i \notin [d - l(w(\nu+\nu_-)), d - l(w(\nu-\nu_+))]$.

For these assertions to hold, we may replace X and $\mathsf{X}^{\mathrm{tor}}$ with their connected components $\Gamma_i \backslash \mathsf{H}_0$ and $(\Gamma_i \backslash \mathsf{H}_0)^{\mathrm{tor}}_{\Sigma_i}$, respectively, replace G with $\widetilde{\mathrm{G}}_0$, replace H with H_0, replace each Γ_i with a neat finite index normal subgroup of its preimage in $\widetilde{\mathrm{G}}_0(\mathbb{Q})$, and replace each Σ_i with a projective and smooth refinement, so that all weights of $\widetilde{\mathrm{M}}_0$ and $\widetilde{\mathrm{G}}_{0,\mathbb{C}}$ can be used for defining automorphic bundles, and so that we may take ν_+ and ν_- here to be the same ν_+ as in Corollary 3.13. (The replacement of Σ_i with a refinement does not change the coherent cohomology, as usual, by the arguments in [29, Ch. I, Sec. 3, especially p. 44, Cor. 2].)

The proof of Theorem 4.1 will be given below, after stating Theorem 4.3.

Remark 4.2. Theorem 4.1 generalizes the previously known results in [35], [36], and [37] in PEL-type cases over $\mathbb{C}$, which were based on techniques developed in positive characteristics in [11], [25], [27], [12], and [43]. (In the Siegel case, similar results also based on techniques developed in positive characteristics were independently discovered in [48] and [49], although the methods there depended on special results that are only available in the Siegel case in the literature.) Our proof of Theorem 4.1 will be based on a rather general vanishing theorem for mixed Hodge modules, recently proved in [50], which is based on Saito's theory in [44] which is complex analytic in nature and have not yet been generalized to positive characteristics. In any case, the rather geometric proofs of Theorem 4.1 and its predecessors have the advantage of not using any techniques based on automorphic forms, and hence do not depend on the as-yet-still-unanswered question of whether cohomology groups like $H^i(\mathsf{X}^{\mathrm{tor}}, \underline{W}_\nu^{\mathrm{can}})$ and $H^i(\mathsf{X}^{\mathrm{tor}}, \underline{W}_\nu^{\mathrm{sub}})$ are represented by automorphic forms (which cannot be deduced from the results of [15] when ν is not cohomological in the sense of Definition 2.6). To the best of our knowledge, Theorem 4.1 is not covered by obvious generalizations of other considerations in the literature.

Theorem 4.3 (Suh; see [50])**.** *Suppose D' is an effective Cartier divisor on $\mathsf{X}^{\mathrm{tor}}$ such that $\mathsf{D}'_{\mathrm{red}} = \mathsf{D}$, and $\mathcal{L}$ is a semiample line bundle such that there exists an integer $N_0 \geq 1$ such that $\mathcal{L}^{\otimes N}(-\mathsf{D}')$ is ample for all $N \geq N_0$. Then, for any irreducible representation $V_{[\mu]}$ of $\mathrm{G}_\mathbb{C}$ as in Section 2.3, we have:*

(1) $H^i\big(\mathsf{X}^{\mathrm{tor}}, \mathcal{L}^{-1} \underset{\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}}}{\otimes} \mathrm{Gr}^{\mathsf{F}}(\mathrm{DR}^\bullet((\underline{V}_{[\mu]}^\vee)^{\mathrm{can}}))\big) = 0$ *for every* $i < d$.

(2) $H^i\big(\mathsf{X}^{\mathrm{tor}}, \mathcal{L} \underset{\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}}}{\otimes} \mathrm{Gr}^{\mathsf{F}}(\mathrm{DR}^\bullet((\underline{V}_{[\mu]}^\vee)^{\mathrm{sub}}))\big) = 0$ *for every* $i > d$.

Proof. Since any $\mathcal{L}$ as in the statement of the theorem is nef and big, and since the local system associated with $(\underline{V}_{[\mu]}^\vee, \nabla)$ has unipotent monodromy (by [1, Ch. III, Sec. 5, Main Thm. I and its proof] and the explanation in [37, Sec. 6.1]), the assertions of the theorem follow from the vanishing results of [50] for canonical extensions of polarized variations of Hodge structures. □

Remark 4.4. When $\mathsf{X}^{\mathrm{tor}}$ is a union of connected components of the complex fiber of some toroidal compactification of a PEL-type Shimura variety (as in [32, Thm. 6.4.1.1 and 7.3.3.4]), Theorem 4.3 follows from [36, Cor. 6.2] and [37, Prop. 7.21], which were based on [25, Cor. 4.16] and [37, Thm. 3.24], respectively. It seems plausible that the methods there (using geometry in good mixed characteristics) can be extended to cover all abelian-type cases, although they have not been carried out yet (as far as we know).

Proof of Theorem 4.1. By Propositions 2.4 and 3.7, the two vanishing statements in Theorem 4.3 imply the following two, for all $\mu \in \mathrm{X}^+_{\mathrm{G}_\mathbb{C}}$ and all $w \in \mathrm{W}^{\mathrm{M}}$:

(1) $H^{i-l(w)}\big(\mathsf{X}^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_{-\nu_-} \underset{\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}}}{\otimes} (\underline{W}^\vee_{w\cdot\mu})^{\mathrm{can}}\big) = 0$ for every $i < d$.

(2) $H^{i-l(w)}\big(\mathsf{X}^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_{\nu_+} \underset{\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}}}{\otimes} (\underline{W}^\vee_{w\cdot\mu})^{\mathrm{sub}}\big) = 0$ for every $i > d$.

Since μ and w are arbitrary, these imply the first two vanishing statements in Theorem 4.1, and hence also the third. (This is the same argument as in [37, Sec. 7.3 and 7.4].) The last paragraph of Theorem 4.1 is self-explanatory. □

4.2. **Higher direct images and higher Koecher's principle.**

Theorem 4.5 (cf. [33, Thm. 3.9 and Rem. 10.1; see also Rem. 3.10])**.** *For every $\nu \in \mathrm{X}^+_{\mathrm{M}}$, we have $R^i \oint_* \underline{W}^{\mathrm{sub}}_\nu = 0$ for all $i > 0$.*

Proof. By the same method as in [34], by (2) of Theorem 4.1, it suffices to show that the analogue of [34, Prop. 2.6] is true, which we can reformulate as follows: By definition of positive parallel weights in Definition 3.1, it suffices to note that there exists some integer N_0 (depending on ν) such that $(\nu + N\rho^{\mathrm{M}}, \alpha^\vee) \geq 0$ for all $\alpha \in \Phi^+_{\mathrm{G}_\mathbb{C}}$ and all $N \geq N_0$. This is because, if $\alpha \in \Phi^+_{\mathrm{M}}$, then $(\nu, \alpha^\vee) \geq 0$ and

$(\rho^{\mathrm{M}}, \alpha^\vee) = 0$ by Lemma 2.2; otherwise $\alpha \in \Phi^{\mathrm{M},+}_{\mathrm{G}_{\mathbb{C}}}$ and $(\rho^{\mathrm{M}}, \alpha^\vee) > 0$ by Lemma 2.3, and therefore it suffices to take $N_0 \geq -(\nu, \alpha^\vee)/(\rho^{\mathrm{M}}, \alpha^\vee)$ for all such α. □

Remark 4.6. While Theorem 4.5 is not new, the proof based on Theorem 4.1 suggests an intriguing relation between vanishing results in rather different contexts.

Theorem 4.7 (higher Koecher's principle; cf. [33, Thm. 2.5 and Rem. 10.1])**.** *Let $\nu \in \mathrm{X}^+_{\mathrm{M}}$. Let $j^{\mathrm{tor}} : \mathsf{X} \hookrightarrow \mathsf{X}^{\mathrm{tor}}$ and $j^{\mathrm{min}} : \mathsf{X} \hookrightarrow \mathsf{X}^{\mathrm{min}}$ denote the canonical morphisms, and let $c_{\mathsf{X}} := \mathrm{codim}(\mathsf{X}^{\mathrm{min}} - \mathsf{X}, \mathsf{X}^{\mathrm{min}})$. Then the canonical morphism*

$$R^i \oint_* \underline{W}^{\mathrm{can}}_\nu \to R^i j^{\mathrm{min}}_* \underline{W}_\nu \tag{4.8}$$

induced by j^{tor} is an isomorphism for all $i < c_{\mathsf{X}} - 1$, and is injective for $i = c_{\mathsf{X}} - 1$.

Consequently, by the Leray spectral sequence [17, Ch. II, Thm. 4.17.1]*, for each open subset U of $\mathsf{X}^{\mathrm{min}}$, the canonical restriction morphism*

$$H^i(\oint^{-1}(U), \underline{W}^{\mathrm{can}}_\nu) \to H^i((j^{\mathrm{min}})^{-1}(U), \underline{W}_\nu) \tag{4.9}$$

is bijective (*resp. injective*) *for all $i < c_{\mathsf{X}} - 1$* (*resp. $i = c_{\mathsf{X}} - 1$*)*.* (*When $i = 0$, $U = \mathsf{X}^{\mathrm{min}}$, and $c_{\mathsf{X}} > 1$, this is the usual Koecher's principle.*)

The analogous statements are true if we replace all varieties and sheaves with their complex analytifications (*with sections represented by holomorphic functions*)*.*

Proof. As explained in [33, Rem. 10.1], the same methods as in [33, Sec. 3–8] also work here. Nevertheless, by the same method based on Serre duality as in [33, Sec. 8], we have a short-cut by using Theorem 4.5 here (with its proof based on Theorem 4.1) instead of [33, Thm. 3.9] there. (Then the reduction of the complex analytic assertion to the algebraic one follows from the same steps as in [33, Sec. 3], based on [18, VIII, Prop. 3.2], [19, XII, Prop. 2.1], and [46, Thm. A, A′, and B].) □

4.3. **Vanishing of de Rham cohomology.**

Theorem 4.10 (cf. [36, Thm. 8.16] and [37, Thm. 8.18])**.** *For each irreducible representation $V_{[\mu]}$ of $\mathrm{G}_{\mathbb{C}}$ such that every $\mu' \in [\mu]$ is* **sufficiently regular** *in the sense that, for each positive coroot $\alpha^\vee$ of $\mathrm{G}_{\mathbb{C}}$, which necessarily comes from some $\mathbb{C}$-simple factor of $\widetilde{\mathrm{G}}_{0,\mathbb{C}}$, we have* (*see Definition* 2.1)*:*

$$(\mu', \alpha^\vee) \geq \begin{cases} 0, & \text{if the factor is compact in that its roots are all compact;} \\ 1, & \text{if the factor is not compact and not of types B or C;} \\ 2, & \text{if the factor is not compact but is of types B or C.} \end{cases} \tag{4.11}$$

Then we have:

(1) $H^i_{\mathrm{dR}}(\mathsf{X}, \underline{V}^\vee_{[\mu]}) := H^i(\mathsf{X}^{\mathrm{tor}}, \mathrm{DR}^\bullet((\underline{V}^\vee_{[\mu]})^{\mathrm{can}}) = 0$ *for every $i < d$.*
(2) $H^i_{\mathrm{dR,c}}(\mathsf{X}, \underline{V}^\vee_{[\mu]}) := H^i(\mathsf{X}^{\mathrm{tor}}, \mathrm{DR}^\bullet((\underline{V}^\vee_{[\mu]})^{\mathrm{sub}}) = 0$ *for every $i > d$.*
(3) $H^i_{\mathrm{dR,int}}(\mathsf{X}, \underline{V}^\vee_{[\mu]}) := \mathrm{image}(H^i_{\mathrm{dR,c}}(\mathsf{X}, \underline{V}^\vee_{[\mu]}) \to H^i_{\mathrm{dR}}(\mathsf{X}, \underline{V}^\vee_{[\mu]})) = 0$ *for every $i \neq d$.*

Proof. We may and we shall perform the replacements as in the last paragraph of Theorem 4.1, so that all weights of $\widetilde{\mathrm{M}}_0$ and $\widetilde{\mathrm{G}}_{0,\mathbb{C}}$ can be used for defining automorphic bundles. By using Hodge spectral sequences, and by Corollary 2.5, it suffices to show that, for all $w \in \mathrm{W}^{\mathrm{M}}$ and all $\mu' \in [\mu]$, we have:

(1) $H^{i-l(w)}\big(\mathsf{X}^{\mathrm{tor}}, (\underline{W}^\vee_{w\cdot\mu'})^{\mathrm{can}}\big) = 0$ for every $i < d$.
(2) $H^{i-l(w)}\big(\mathsf{X}^{\mathrm{tor}}, (\underline{W}^\vee_{w\cdot\mu'})^{\mathrm{sub}}\big) = 0$ for every $i > d$.

By Theorem 4.1, it suffices to show that there exists a positive parallel weight ν_+ as in Definition 3.1 such that, for all $w \in \mathrm{W}^{\mathrm{M}}$ and all $\mu' \in [\mu]$, the weights $w \cdot (\mu' \pm \nu_+) = w \cdot (\mu' \pm w^{-1}(\nu_+))$ in X_{M} are of the form $w \cdot \mu''_\pm$ for some weights $\mu''_\pm$ in $\mathrm{X}^+_{\mathrm{G}_{\mathbb{C}}}$ (cf. Definition 2.6), or (equivalently) such that

$$(\mu' \pm w^{-1}(\nu_+), \alpha^\vee) \geq 0 \tag{4.12}$$

for all simple $\alpha \in \Phi^+_{\mathrm{G}_{\mathbb{C}}}$. Since every $\mu' \in [\mu]$ satisfies (4.11), it suffices to show that there exists a positive parallel weight ν_+ such that, for all $w \in \mathrm{W}^{\mathrm{M}}$ and all simple $\alpha \in \Phi^+_{\mathrm{G}_{\mathbb{C}}}$, where α comes from some $\mathbb{C}$-simple factor of $\widetilde{\mathrm{G}}_{0,\mathbb{C}}$, we have

$$|(w^{-1}(\nu_+), \alpha^\vee)| \leq \begin{cases} 0, & \text{if the factor is compact;} \\ 1, & \text{if the factor is not of types B or C;} \\ 2, & \text{in all cases.} \end{cases} \tag{4.13}$$

Equivalently, it suffices to show that there exists a positive parallel weight ν_+ such that, for all $w \in \mathrm{W}^{\mathrm{M}}$ and all (not necessarily positive simple) $\alpha \in \Phi_{\mathrm{G}_{\mathbb{C}}}$, where α comes from some $\mathbb{C}$-simple factor of $\widetilde{\mathrm{G}}_{0,\mathbb{C}}$, we have

$$|(\nu_+, \alpha^\vee)| \leq \begin{cases} 0, & \text{if the factor is compact;} \\ 1, & \text{if the factor is not of types B or C;} \\ 2, & \text{in all cases.} \end{cases} \tag{4.14}$$

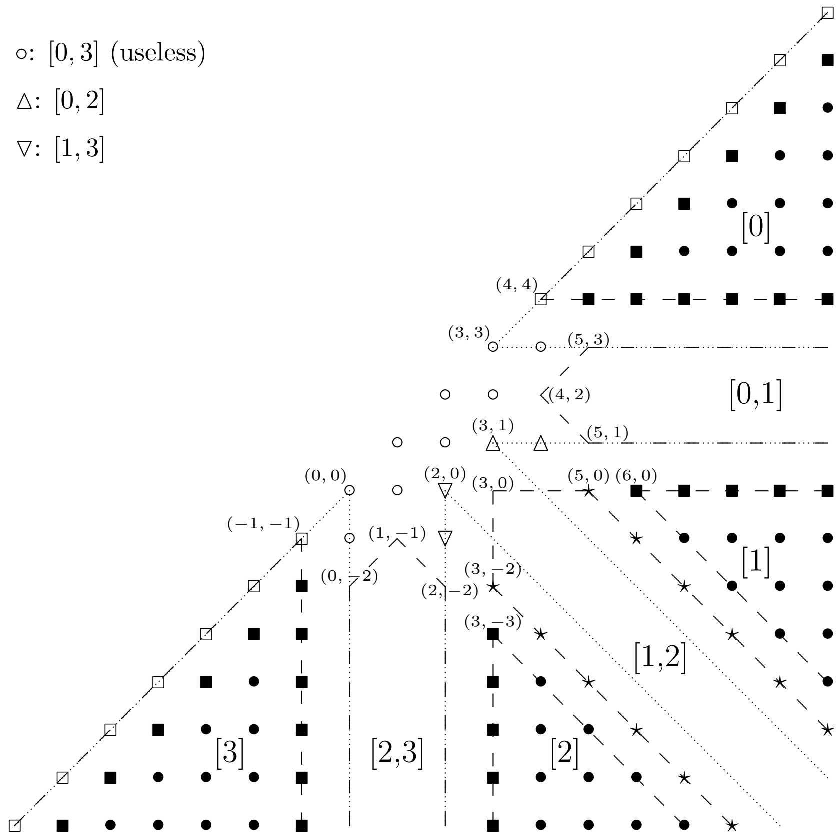
Then the existence of such a ν_+ follows from Corollary 3.13, as desired. (This is the same argument as in the proofs of [36, Thm. 8.16] and [37, Thm. 8.18].) □

Remark 4.15. When none of the simple factors of $\widetilde{\mathrm{G}}_{0,\mathbb{C}}$ is of types B or C, the sufficient regularity condition in Theorem 4.10 is no stronger than the usual regularity condition. In particular, even in PEL-type cases, Theorem 4.10 slightly improves [36, Thm. 8.16] and [37, Thm. 8.18] (when there are some factors of type D).

Remark 4.16. When X is compact, the simplest proof of Theorem 4.10 (assuming only that every $\mu' \in [\mu]$ is regular) is in [13, Sec. 5, Cor. of Thm. 7], by using C^∞-resolutions of vector bundles and harmonic forms. It also follows from the more powerful results of [51], which also work for non-Hermitian locally symmetric spaces. When X is noncompact, by using mixed Hodge theory as in [14, Ch. VI, Sec. 5] and [22, Cor. 4.2.3] to show that Faltings's dual BGG spectral sequences as in Proposition 2.4 degenerate, in the adelic setting, Theorem 4.10 (assuming only that every $\mu' \in [\mu]$ is regular) also follows from [38, Cor. 5.6]. Nevertheless, our proof of Theorem 4.10 here is based on Theorem 4.1 (see Remark 4.2) and the rather combinatorial Theorem 3.8, which are logically independent of the consideration of automorphic forms as in [38, Cor. 5.6].

4.4. **Illustrative examples of low ranks.** To better understand Theorem 4.1 (and implicitly, also Theorem 4.10), let us include some illustrative examples of low ranks (which can be practically plotted in two dimensions), although they have already shown up in the results in the PEL-type case in [36] and [37]. (Nevertheless, they provide examples of the results of [36] and [37] even for torsion coefficients, which might be of some independent interest.)

Example 4.17 (Siegel modular threefolds). Let us adopt the notation system in Section 3.3.3, with $n = 2$. Then the vanishing given by Theorem 4.1 can be visualized as follows: (The positive parallel weights are of the form $k(1, 1)$ for $k \in \mathbb{Z}_{\geq 1}$.)



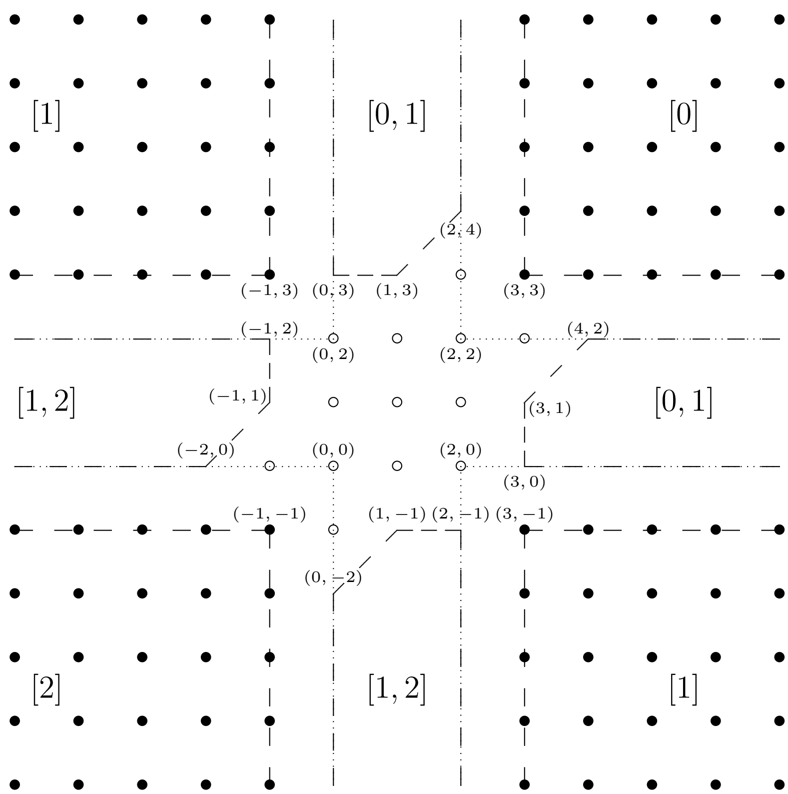
The four chambers whose walls are formed by (partially) dotted half-lines, with vertices at $(0,0)$, $(2,0)$, $(3,1)$, and $(3,3)$, are the chambers for cohomological weights. (Note that we have $\Omega^0_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D}) \cong \underline{W}^{\mathrm{can}}_{(0,0)}$, $\Omega^1_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D}) \cong \underline{W}^{\mathrm{can}}_{(2,0)}$, $\Omega^2_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D}) \cong \underline{W}^{\mathrm{can}}_{(3,1)}$, and $\Omega^3_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D}) \cong \underline{W}^{\mathrm{can}}_{(3,3)}$. In this case, all the elements in W^{M} happen to have different lengths.) The seven regions with boundaries given by dashed line segments and half-lines, which are marked in their interiors by intervals $[a,b]$ or rather $[a] = [a,a]$, are the regions (including their boundaries) for weights $\nu = (k_1,k_2)$ with coordinates (k_1,k_2) such that:

(1) $H^i(\mathsf{X}^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_{\nu}) = 0$ for all $i < a$;
(2) $H^i(\mathsf{X}^{\mathrm{tor}}, \underline{W}^{\mathrm{sub}}_{\nu}) = 0$ for all $i > b$; and
(3) $H^i_{\mathrm{int}}(\mathsf{X}^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_{\nu}) = 0$ for all $i \notin [a,b]$.

The two weights $(3,1)$ and $(4,1)$ denoted as $\triangle$ means $[a,b] = [0,2]$ in the above sense; the two weights $(2,0)$ and $(2,-1)$ denoted as $\triangledown$ means $[a,b] = [1,3]$ in the above sense; and the nine weights denoted as $\circ$ means $[a,b] = [0,3]$, which are unfortunately useless because they provide no information concerning the vanishing for the coherent cohomology of threefolds. The weights denoted by $\bullet$ are the weights appearing in the Hodge cohomology as in Corollary 2.5 for those $[\mu]$ for which the sufficiently regularity condition in Theorem 4.10 holds. The weights denoted by $\blacksquare$ and $\square$ are the other ones such that Theorem 4.1 implies that the corresponding interior cohomology is concentrated in just one degree for each of them. The two colors ■ and □ are used for regular and irregular weights, respectively. For the weights denoted by $\star$, which are along the half-lines starting from $(3,-2)$ and $(5,0)$ in the direction of $(1,-1)$, they are regular and the corresponding interior cohomology is also concentrated in just one degree, by [38, Cor. 5.6] and [22, Cor. 4.2.3]. But our method fails to detect such stronger vanishing. This is a defect of our method when there are factors of types B and C.

Example 4.18 (Hilbert modular surfaces). Suppose $\widetilde{\mathrm{G}}_0$ is isomorphic to the restriction of scalar $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{SL}_2$ for some real quadratic extension F of $\mathbb{Q}$. Let us adopt the notation system in Section 3.3.3, with $n=1$, but with the root system doubled because there are two $\mathbb{C}$-simple factors in the same $\mathbb{Q}$-simple factor. Then the vanishing given by Theorem 4.1 can be visualized as follows: (The positive parallel weights are of the form $k(1,1)$ for $k \in \mathbb{Z}_{\geq 1}$.)

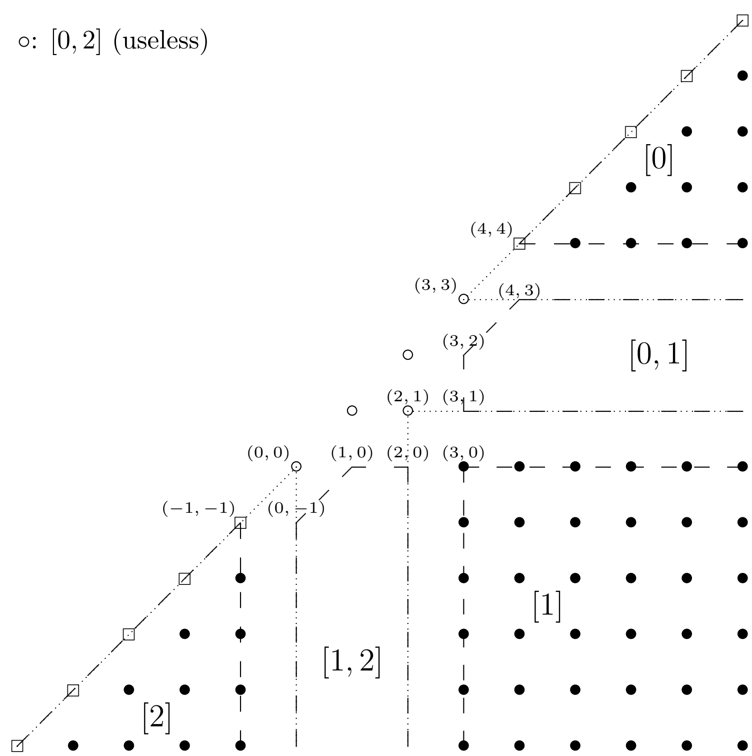
$\circ$: $[0,2]$ (useless)



The four chambers whose walls are formed by (partially) dotted half-lines, with vertices at $(0,0)$, $(2,0)$, $(0,2)$, and $(2,2)$, are the chambers for cohomological weights. (Note that we have $\Omega^0_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D}) \cong \underline{W}^{\mathrm{can}}_{(0,0)}$, $\Omega^1_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D}) \cong \underline{W}^{\mathrm{can}}_{(2,0)} \oplus \underline{W}^{\mathrm{can}}_{(0,2)}$, and $\Omega^2_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D}) \cong \underline{W}^{\mathrm{can}}_{(2,2)}$. In this case, two of the elements in W^{M} have the same length.) The eight regions with boundaries given by dashed line segments and half-lines, which are marked in their interiors by intervals $[a,b]$ or rather $[a]=[a,a]$, have a similar meaning as in Example 4.17. The thirteen weights denoted as $\circ$

means $[a,b] = [0,2]$, which are unfortunately useless because they provide no information concerning the vanishing for the coherent cohomology of surfaces. The weights denoted by $\bullet$, which are all the regular ones, have a similar meaning as in Example 4.17, although we have no weights here that should be denoted by $\blacksquare$, $\square$, or $\star$. Note that, while $(\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{SL}_2)_{\mathbb{C}} \cong \mathrm{SL}_{2,\mathbb{C}} \times \mathrm{SL}_{2,\mathbb{C}}$ when F is totally real quadratic over $\mathbb{Q}$, the vanishing results are not the Künneth products (in the obvious sense, by summing up the vanishing degrees) of the corresponding ones for SL_2. (We will see similar phenomena in Examples 5.31, 5.38, 5.41, and 5.49 below.)

Example 4.19 (Picard modular surfaces). Let us adopt the notation system in Section 3.3.1, with $n = 2$ and $r = 1$. For simplicity, we shall plot any weight (k_1, k_2, k_3) mod $(1,1,1)$ as $(k_1 - k_3, k_2 - k_3)$. Then the vanishing given by Theorem 4.1 can be visualized as follows: (Up to a multiple of $(1,1,1)$, and up to writing any weight (k_1, k_2, k_3) mod $(1,1,1)$ as $(k_1 - k_3, k_2 - k_3)$ as above, the positive parallel weights are of the form $k(1,1)$ for $k \in \mathbb{Z}_{\geq 1}$. Of course, the following figure has "wrong angles" because it is a projection.)



The three chambers whose walls are formed by (partially) dotted half-lines, with vertices at $(0,0)$, $(2,1)$, and $(3,3)$, are the chambers for cohomological weights. (Note that we have $\Omega^0_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D}) \cong \underline{W}^{\mathrm{can}}_{(0,0)}$, $\Omega^1_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D}) \cong \underline{W}^{\mathrm{can}}_{(2,1)}$, and $\Omega^2_{\mathsf{X}^{\mathrm{tor}}/\mathbb{C}}(\log \mathsf{D}) \cong \underline{W}^{\mathrm{can}}_{(3,3)}$. In this case, again, all elements in W^{M} happen to have different lengths.) The five regions with boundaries given by dashed line segments and half-lines, which are marked in their interiors by intervals $[a,b]$ or rather $[a] = [a,a]$, have a similar meaning as in Example 4.17. The five weights denoted as $\circ$ means $[a,b] = [0,2]$, which are unfortunately useless because they provide no information concerning the vanishing for the coherent cohomology of surfaces. The weights denoted by •, which are all the regular ones, have a similar meaning as in Example 4.17. We also have the weights denoted by □, which are irregular, but Theorem 4.1 implies that the corresponding interior cohomology is still concentrated in just one degree for each of them. We have no weights here that should be denoted by ■ or ⋆ as in Example 4.17.

5. Algorithms for determining degrees of vanishing

In this section, we record some explicit algorithms for determining the degrees of vanishing in Theorem 4.1, which are important for practical applications. Given any weight $\nu \in \mathrm{X}^+_{\mathrm{M}}$, we need to find positive parallel weights ν_+ and ν_- such that $\nu+\nu_+$ and $\nu+\nu_-$ are both cohomological, and such that the interval $[d-l(w(\nu+\nu_-)), d-l(w(\nu-\nu_+))]$ is as short as possible. Since the definition of positive parallel weights depends only on the pullback of the weight to the $\mathbb{Q}$-simple factors of $\widetilde{\mathrm{G}}_0$, since the dimension d of H_0 is the length of the longest element in W^{M}, and since the length of any $w \in \mathrm{W}_{\mathrm{G}_{\mathbb{C}}}$ is the sum of the lengths of the pullbacks of w to the $\mathbb{C}$-simple factors of $\widetilde{\mathrm{G}}_{0,\mathbb{C}}$, we may assume that G is semisimple and $\mathbb{Q}$-simple, and that $\mathrm{G}_{\mathbb{C}}$ is connected and simply connected. (That is, we shall first compute the vanishing degrees over the $\mathbb{Q}$-simple factors of $\widetilde{\mathrm{G}}_0$, and sum them up afterwards.)

In what follows, for each $\nu \in \mathrm{X}^+_{\mathrm{M}}$, each of our algorithms will produce an interval $[d^-, d^+]$, which have the same meaning as the intervals in Example 4.17: (i) $H^i(\mathsf{X}^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_\nu) = 0$ for all $i < d^-$; (ii) $H^i(\mathsf{X}^{\mathrm{tor}}, \underline{W}^{\mathrm{sub}}_\nu) = 0$ for all $i > d^+$; and (iii) $H^i_{\mathrm{int}}(\mathsf{X}^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_\nu) = 0$ for all $i \notin [d^-, d^+]$. (As explained above, if there are more than one $\mathbb{Q}$-simple factors, the ends of the intervals need to be summed up.)

We shall adopt the notation system as in Section 3.3, with an additional υ in the beginning of the subscripts, such as $\alpha_{\upsilon,1}, \alpha_{\upsilon,2}, \ldots$, for each $\upsilon \in \Upsilon$, indicating the $\mathbb{C}$-simple factor to which the objects belong.

The overall strategy can be summarized as follows. Suppose $\nu \in \mathrm{X}^+_{\mathrm{M}}$, which is of the form $\nu = (\nu_\upsilon)_{\upsilon\in\Upsilon}$, where $\nu_\upsilon \in \mathrm{X}^+_{\mathrm{M}_\upsilon}$ for all $\upsilon \in \Upsilon$.

Step 1. Switch from ν to the *dual representation weight*, namely the weight $\lambda = (\lambda_\upsilon)_{\upsilon\in\Upsilon} \in \mathrm{X}^+_{\mathrm{M}}$ such that $W_\nu \cong W^\vee_\lambda$. (The methods for writing down such dual weights will be explained in Section 5.1 below.)

Step 2. For each integer $s \in \mathbb{Z}$, consider $\lambda^{(s)} = (\lambda^{(s)}_\upsilon)_{\upsilon\in\Upsilon}$ with

$$\lambda^{(s)}_\upsilon := \lambda_\upsilon + \rho_{\mathrm{G}_\upsilon} + s\varpi_{\upsilon,0}, \tag{5.1}$$

where $\varpi_{\upsilon,0}$ is the fundamental weight dual to the simple positive root α_0 such that $\alpha_0 \notin \Phi_{\mathrm{M}_\upsilon}$. (We set $\varpi_{\upsilon,0}$ to be zero if no such α_0 exists, which is the case when $\mathrm{M}_\upsilon = \mathrm{P}_\upsilon = \mathrm{G}_\upsilon$.)

Step 3. For each $\upsilon \in \Upsilon$, we say that $\lambda^{(s)}_\upsilon$ is regular if it does not lie on the walls of the Weyl chambers of the weights of $\mathrm{X}_{\mathrm{G}_\upsilon}$. We say that $\lambda^{(s)} = (\lambda^{(s)}_\upsilon)_{\upsilon\in\Upsilon}$ is regular if $\lambda^{(s)}_\upsilon$ is regular for all $\upsilon \in \Upsilon$. (This is equivalent to saying that $\lambda^{(s)} - \rho_{\mathrm{G}_{\mathbb{C}}} = w \cdot \mu$ for some $w \in \mathrm{W}^{\mathrm{M}}$ and $\mu \in \mathrm{X}^+_{\mathrm{G}_{\mathbb{C}}}$; cf. Definition 2.6.)

To each regular weight $\lambda^{(s)}_\upsilon$, we attach the unique weight $\kappa^{(s)}_\upsilon$ in the same Weyl chamber that is the conjugation of $\rho_{\mathrm{G}_\upsilon}$ by some element $w^{(s)}_\upsilon$ in W^{M}. Then we define

$$l^{(s)}_\upsilon := l(w^{(s)}_\upsilon) \tag{5.2}$$

and

$$d_v^{(s)} := d_v - l_v^{(s)}, \tag{5.3}$$

where

$$d_v = \dim_{\mathbb{C}}(\mathrm{G}_v) - \dim_{\mathbb{C}}(\mathrm{P}_v) = \tfrac{1}{2}(\dim_{\mathbb{C}}(\mathrm{G}_v) - \dim_{\mathbb{C}}(\mathrm{M}_v)). \tag{5.4}$$

(The methods for effectively determining the regularity of $\lambda_v^{(s)}$ and the corresponding value of $l_v^{(s)}$ will be explained in Section 5.2 below.)

Step 4. Compute with $s = 1, 2, \ldots$ and take s_+ to be first value of such an s such that $\lambda^{(s_+)} = (\lambda_v^{(s_+)})_{v\in\Upsilon}$ is regular. Similarly, compute with $s = -1, -2, \ldots$ and take s_- to be the first value of such an s such that $\lambda^{(s_-)} = (\lambda_v^{(s_-)})_{v\in\Upsilon}$ is regular. Then we define

$$d^+ := d^{(s_+)} := \sum_{v\in\Upsilon} d_v^{(s_+)} \tag{5.5}$$

and

$$d^- := d^{(s_-)} := \sum_{v\in\Upsilon} d_v^{(s_-)}. \tag{5.6}$$

The resulted interval $[d^-, d^+]$ is what we want.

Remark 5.7. The strategy we present here also apply to the results in [36] and [37], provided that the weights are p-small in the senses required there, except that for factors of type D (which is necessarily of type $\mathrm{D}_n^{\mathbb{H}}$ for some n), we need to shift by $2\varpi_{v,0} = (1, 1, \ldots, 1)$ instead of $\varpi_{v,0} = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, because this is the smallest positive parallel weight allowed in the context of [36] and [37].

5.1. **Dual weights.** While the general principle is simple—take the longest Weyl element w_0 of $\mathrm{W}_{\mathrm{M}_v}$, and map $\nu_v \in \mathrm{X}_{\mathrm{M}_v}^+$ to $\lambda_v = -w_0(\nu)$—let us nevertheless spell out the explicit changes of coordinates using the notation system in Section 3.3.

5.1.1. *Type* A. Suppose we are in the context of Section 3.3.1, with some r_v such that $1 \leq r_v \leq n_v$. Then we map the weight $\nu_v = (\nu_{v,1}, \nu_{v,2}, \ldots, \nu_{v,r_v}; \nu_{v,r_v+1}, \nu_{v,r_v+2}, \ldots, \nu_{v,n_v+1})$ in $\mathrm{X}_{\mathrm{M}_v}$ to $\lambda_v = (-\nu_{v,r_v}, -\nu_{v,r_v-1}, \ldots, -\nu_{v,1}; -\nu_{v,n_v+1}, -\nu_{v,n_v}, \ldots, -\nu_{v,r_v+1})$. When no r_v exists, in which case $\mathrm{X}_{\mathrm{M}_v} = \mathrm{X}_{\mathrm{G}_v}$, we apply this recipe with $r_v = 0$ or $n_v + 1$.

5.1.2. *Type* B. Suppose we are in the context of Section 3.3.2, with $r_v = 1$. Then we map the weight $\nu_v = (\nu_{v,1}; \nu_{v,2}, \ldots, \nu_{v,n_v})$ in $\mathrm{X}_{\mathrm{M}_v}$ to $\lambda_v = (-\nu_{v,1}; \nu_{v,2}, \ldots, \nu_{v,n_v})$, changing only the sign of the first entry $\nu_{v,1}$. When no r_v exists, in which case $\mathrm{X}_{\mathrm{M}_v} = \mathrm{X}_{\mathrm{G}_v}$, we have $\lambda_v = \nu_v$, with exactly the same entries.

5.1.3. *Type* C. Suppose we are in the context of Section 3.3.3, with $r_v = n_v$. Then we map the weight $\nu_v = (\nu_{v,1}, \nu_{v,2}, \ldots, \nu_{v,n_v})$ in $\mathrm{X}_{\mathrm{M}_v}$ to $\lambda_v = (-\nu_{v,n_v}, -\nu_{v,n_v-1}, \ldots, -\nu_{v,1})$. When no r_v exists, in which case $\mathrm{X}_{\mathrm{M}_v} = \mathrm{X}_{\mathrm{G}_v}$, we have $\lambda_v = \nu_v$, with exactly the same entries, as in the type B case above.

5.1.4. *Type* D. Suppose we are in the context of Section 3.3.4, with $n_v \geq 4$ and $r_v = 1$, $n_v - 1$, or n_v. If $r_v = 1$, then we map the weight $\nu_v = (\nu_{v,1}; \nu_{v,2}, \ldots, \nu_{v,n_v})$ in X_{M_v} to $\lambda_v = (-\nu_{v,1}; \nu_{v,2}, \ldots, \nu_{v,n_v-1}, (-1)^{n_v-1}\nu_{v,n_v})$, where the sign of the first entry $\nu_{v,1}$ is changed as in the type B case above, and where the sign of the last entry ν_{v,n_v} is changed exactly when n_v is even. If $r_v = n_v - 1$, then we map the weight $\nu_v = (\nu_{v,1}, \nu_{v,2}, \ldots, \nu_{v,n_v})$ in X_{M_v} to $\lambda_v = (\nu_{v,n_v}, -\nu_{v,n_v-1}, \ldots, -\nu_{v,2}, \nu_{v,1})$, which differ from the type C case above by the signs at the first and the n_v-th terms. If $r_v = n_v$, then we map the weight $\nu_v = (\nu_{v,1}, \nu_{v,2}, \ldots, \nu_{v,n_v})$ in X_{M_v} to $\lambda_v = (-\nu_{v,n_v}, -\nu_{v,n_v-1}, \ldots, -\nu_{v,1})$ as in the type C case above. When no r_v exists, in which case $X_{M_v} = X_{G_v}$, we map the weight $\nu_v = (\nu_{v,1}, \nu_{v,2}, \ldots, \nu_{v,n_v})$ to $\nu_v = (\nu_{v,1}, \nu_{v,2}, \ldots, \nu_{v,n_v-1}, (-1)^{n_v}\nu_{v,n_v})$, where the sign of the last entry ν_{v,n_v} is changed exactly when n_v is odd.

5.1.5. *Type* E_6. Suppose we are in the context of Section 3.3.5, with $r_v = 1$ or 6. Then we map the weight ν_v in X_{M_v} to the weight $\lambda_v = \nu_v T_v$ (as row vectors), where

$$T_v = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{5.8}$$

depending on whether $r_v = 1$ or 6. In both cases, T_v maps $\varpi_{v,0}$ to $-\varpi_{v,0}$. On the orthogonal complement of $\varpi_{v,0}$, it swaps the two roots $\alpha_{v,2}$ and $\alpha_{v,4}$ (resp. $\alpha_{v,4}$ and $\alpha_{v,5}$) in the first (resp. second) case, while preserving each of the other roots. When no r_v exists, in which case $X_{M_v} = X_{G_v}$, we map the weight ν_v to the weight $\lambda_v = \nu_v T_v$, with

$$T_v = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \tag{5.9}$$

which swaps the pair of roots $\alpha_{v,1}$ and $\alpha_{v,6}$, and also the pair of roots $\alpha_{v,2}$ and $\alpha_{v,5}$, while preserving each of $\alpha_{v,3}$ and $\alpha_{v,4}$.

5.1.6. *Type* E_7. Suppose we are in the context of Section 3.3.6, with $r_v = 1$. Similar to the type E_6 case above, we map the weight ν_v in X_{M_v} to the weight $\lambda_v = \nu_v T_v$, where

$$T_v = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.10}$$

Again, this matrix T_υ maps the fundamental weight $\varpi_{\upsilon,0}$ to $-\varpi_{\upsilon,0}$. On the orthogonal complement of $\varpi_{\upsilon,0}$, it swaps the pair of roots $\alpha_{\upsilon,2}$ and $\alpha_{\upsilon,7}$, and also the pair of roots $\alpha_{\upsilon,3}$ and $\alpha_{\upsilon,6}$, while preserving each of $\alpha_{\upsilon,4}$ and $\alpha_{\upsilon,5}$. When no r_υ exists, in which case $\mathrm{X}_{\mathrm{M}_\upsilon} = \mathrm{X}_{\mathrm{G}_\upsilon}$, we have $\lambda_\upsilon = \nu_\upsilon$ as in the type B and C cases above.

5.2. **Regularity and Weyl lengths.** In this subsection, we shall assume that $\mathrm{M}_\upsilon \neq \mathrm{G}_\upsilon$ and so that $\mathrm{W}^{\mathrm{M}_\upsilon}$ is nontrivial and some r_υ exists. (Otherwise we can just set $l_\upsilon^{(s)} = 0$ and $d_\upsilon^{(s)} = 0$ in the contexts of (5.2) and (5.3).)

5.2.1. *Type* A. Suppose we are in the context of Section 3.3.1. Then $\lambda_\upsilon^{(s)} = (\lambda_{\upsilon,1}^{(s)}, \lambda_{\upsilon,2}^{(s)}, \dots, \lambda_{\upsilon,n_\upsilon+1}^{(s)})$ is regular if and only if all the values $\lambda_{\upsilon,1}^{(s)}, \lambda_{\upsilon,2}^{(s)}, \dots, \lambda_{\upsilon,n_\upsilon+1}^{(s)}$ are mutually distinct from each others. For each regular $\lambda_\upsilon^{(s)}$, we sort out the values of $\{\lambda_{\upsilon,i}^{(s)}\}_{1\leq i\leq n_\upsilon+1}$ in increasing order such that

$$\lambda_{\upsilon,i_1}^{(s)} < \lambda_{\upsilon,i_2}^{(s)} < \cdots < \lambda_{\upsilon,i_{n_\upsilon+1}}^{(s)}. \tag{5.11}$$

Then we define

$$\kappa_{\upsilon,i_j}^{(s)} := -\tfrac{n_\upsilon}{2} + (j-1) \tag{5.12}$$

for $1 \leq j \leq n_\upsilon + 1$, and define

$$l_\upsilon = (\rho_{\mathrm{G}_\upsilon} - \kappa_\upsilon^{(s)}, \varpi_{\upsilon,0}) = \tfrac{1}{2}d_\upsilon - \sum_{1\leq i\leq r_\upsilon} \kappa_{\upsilon,i}^{(s)} = \tfrac{1}{2}d_\upsilon + \sum_{r_\upsilon < i\leq n_\upsilon+1} \kappa_{\upsilon,i}^{(s)}, \tag{5.13}$$

where $d_\upsilon = r_\upsilon(n_\upsilon + 1 - r_\upsilon) = \sum\limits_{1\leq i\leq r_\upsilon} (n_\upsilon + 2 - 2i) = -\sum\limits_{r_\upsilon < i\leq n_\upsilon+1} (n_\upsilon + 2 - 2i)$. (There is a unique $w_\upsilon^{(s)} \in \mathrm{W}^{\mathrm{M}_\upsilon}$ mapping $\rho_{\mathrm{G}_\upsilon} = \frac{1}{2}(n_\upsilon, n_\upsilon - 2, \dots, 2 - n_\upsilon, -n_\upsilon)$ to $\kappa_\upsilon^{(s)} = (\kappa_{\upsilon,1}^{(s)}, \dots, \kappa_{\upsilon,n_\upsilon+1}^{(s)})$.) Therefore,

$$d_\upsilon^{(s)} = d_\upsilon - l_\upsilon^{(s)} = (\rho_{\mathrm{G}_\upsilon} + \kappa_\upsilon^{(s)}, \varpi_{\upsilon,0}) = \tfrac{1}{2}d_\upsilon + \sum_{1\leq i\leq r_\upsilon} \kappa_{\upsilon,i}^{(s)} = \tfrac{1}{2}d_\upsilon - \sum_{r_\upsilon < i\leq n_\upsilon+1} \kappa_{\upsilon,i}^{(s)}. \tag{5.14}$$

5.2.2. *Type* B. Suppose we are in the context of Section 3.3.2. Then $\lambda_\upsilon^{(s)} = (\lambda_{\upsilon,1}^{(s)}, \lambda_{\upsilon,2}^{(s)}, \dots, \lambda_{\upsilon,n_\upsilon}^{(s)})$ is regular if and only if all the absolute values $|\lambda_{\upsilon,1}^{(s)}|, |\lambda_{\upsilon,2}^{(s)}|, \dots, |\lambda_{\upsilon,n_\upsilon}^{(s)}|$ are nonzero and are mutually distinct from each others. For each regular $\lambda_\upsilon^{(s)}$, we sort out the values of $\{|\lambda_{\upsilon,i}^{(s)}|\}_{1\leq i\leq n_\upsilon}$ in increasing order such that

$$0 < |\lambda_{\upsilon,i_1}^{(s)}| < |\lambda_{\upsilon,i_2}^{(s)}| < \cdots < |\lambda_{\upsilon,i_{n_\upsilon}}^{(s)}|. \tag{5.15}$$

Then we define

$$\kappa_{\upsilon,i_j}^{(s)} := \mathrm{sign}(\lambda_{\upsilon,i_j}^{(s)}) \cdot \tfrac{2j-1}{2} \tag{5.16}$$

for $1 \leq j \leq n_\upsilon$, and define

$$l_\upsilon^{(s)} := (\rho_{\mathrm{G}_\upsilon} - \kappa_\upsilon^{(s)}, \varpi_{\upsilon,0}) = \tfrac{1}{2}d_\upsilon - \kappa_{\upsilon,1}^{(s)}, \tag{5.17}$$

where $d_\upsilon = 2n_\upsilon - 1$. (There is a unique $w_\upsilon^{(s)} \in \mathrm{W}^{\mathrm{M}_\upsilon}$ mapping $\rho_{\mathrm{G}_\upsilon} = \frac{1}{2}(2n_\upsilon - 1, 2n_\upsilon - 3, \dots, 3, 1)$ to $\kappa_\upsilon^{(s)} = (\kappa_{\upsilon,1}^{(s)}, \dots, \kappa_{\upsilon,n_\upsilon+1}^{(s)})$.) Therefore,

$$d_\upsilon^{(s)} = d_\upsilon - l_\upsilon^{(s)} = (\rho_{\mathrm{G}_\upsilon} + \kappa_\upsilon^{(s)}, \varpi_{\upsilon,0}) = \tfrac{1}{2}d_\upsilon + \kappa_{\upsilon,1}^{(s)}. \tag{5.18}$$

5.2.3. *Type* C. Suppose we are in the context of Section 3.3.3. Then $\lambda_v^{(s)} = (\lambda_{v,1}^{(s)}, \lambda_{v,2}^{(s)}, \ldots, \lambda_{v,n_v}^{(s)})$ is regular if and only if all the absolute values $|\lambda_{v,1}^{(s)}|, |\lambda_{v,2}^{(s)}|, \ldots, |\lambda_{v,n_v}^{(s)}|$ are nonzero and are mutually distinct from each others. For each regular $\lambda_v^{(s)}$, we sort out the values of $\{|\lambda_{v,i}^{(s)}|\}_{1\leq i\leq n_v}$ in increasing order such that

$$0 < |\lambda_{v,i_1}^{(s)}| < |\lambda_{v,i_2}^{(s)}| < \cdots < |\lambda_{v,i_{n_v}}^{(s)}|. \tag{5.19}$$

Then we define

$$\kappa_{v,i_j}^{(s)} := \mathrm{sign}(\lambda_{v,i_j}^{(s)}) \cdot j \tag{5.20}$$

for $1 \leq j \leq n_v$, and define

$$l_v^{(s)} := \tfrac{1}{2}(\rho_{\mathrm{G}_v} - \kappa_v^{(s)}, \varpi_{v,0}) = \tfrac{1}{2}\Big(d_v - \sum_{1\leq i\leq n_v} \kappa_{v,i}^{(s)}\Big), \tag{5.21}$$

where $d_v = \frac{1}{2}n_v(n_v+1) = \sum_{1\leq i\leq n_v}(n_v+1-i)$. (There is a unique $w_v^{(s)} \in \mathrm{W}_{\mathrm{G}_v}$ mapping $\rho_{\mathrm{G}_v} = (n_v, n_v-1, \ldots, 2, 1)$ to $\kappa_v^{(s)} = (\kappa_{v,1}^{(s)}, \ldots, \kappa_{v,n_v+1}^{(s)})$.) Therefore,

$$d_v^{(s)} = d_v - l_v^{(s)} = \tfrac{1}{2}(\rho_{\mathrm{G}_v} + \kappa_v^{(s)}, \varpi_{v,0}) = \tfrac{1}{2}\Big(d_v + \sum_{1\leq i\leq n_v} (\kappa_{v,i}^{(s)})\Big). \tag{5.22}$$

5.2.4. *Type* D. Suppose we are in the context of Section 3.3.4. Then $\lambda_v^{(s)} = (\lambda_{v,1}^{(s)}, \lambda_{v,2}^{(s)}, \ldots, \lambda_{v,n_v}^{(s)})$ is regular if and only if all the absolute values $|\lambda_{v,1}^{(s)}|, |\lambda_{v,2}^{(s)}|, \ldots, |\lambda_{v,n_v}^{(s)}|$ are mutually distinct from each others. For each regular $\lambda_v^{(s)}$, we sort out the values of $\{|\lambda_{v,i}^{(s)}|\}_{1\leq i\leq n_v}$ in increasing order such that

$$|\lambda_{v,i_1}^{(s)}| < |\lambda_{v,i_2}^{(s)}| < \cdots < |\lambda_{v,i_{n_v}}^{(s)}|. \tag{5.23}$$

Then we define

$$\kappa_{v,i_j}^{(s)} := \mathrm{sign}(\lambda_{v,i_j}^{(s)}) \cdot (j-1) \tag{5.24}$$

for $1 \leq j \leq n_v$, and define

$$l_v^{(s)} := (\rho_{\mathrm{G}_v} - \kappa_v^{(s)}, \varpi_{v,0}) = \tfrac{1}{2}d_v - (\kappa_v^{(s)}, \varpi_{v,0}) \tag{5.25}$$

where

$$d_v = \begin{cases} 2n_v - 2, & \text{if } r_v = 1; \\ \frac{1}{2}n_v(n_v-1) = \sum_{1\leq i\leq n_v-1}(n_v - i), & \text{if } r_v = n_v - 1 \text{ or } n_v; \end{cases} \tag{5.26}$$

and where

$$(\kappa_v^{(s)}, \varpi_{v,0}) = \begin{cases} \kappa_{v,1}^{(s)}, & \text{if } r_v = 1; \\ \Big(\frac{1}{2}\sum_{1\leq i\leq n_v-1}\kappa_{v,i}^{(s)}\Big) - \frac{1}{2}\kappa_{v,n_v}^{(s)}, & \text{if } r_v = n_v - 1; \\ \frac{1}{2}\sum_{1\leq i\leq n_v}\kappa_{v,i}^{(s)}, & \text{if } r_v = n_v. \end{cases} \tag{5.27}$$

(There is a unique $w_v^{(s)} \in \mathrm{W}_{\mathrm{G}_v}$ mapping $\rho_{\mathrm{G}_v} = (n_v-1, n_v-2, \ldots, 1, 0)$ to $\kappa_v^{(s)} = (\kappa_{v,1}^{(s)}, \ldots, \kappa_{v,n_v+1}^{(s)})$.) Therefore,

$$d_v^{(s)} = d_v - l_v^{(s)} = (\rho_{\mathrm{G}_v} + \kappa_v^{(s)}, \varpi_{v,0}) = \tfrac{1}{2}d_v + (\kappa_v^{(s)}, \varpi_{v,0}). \tag{5.28}$$

5.2.5. *Type* E_6. Suppose we are in the context of Section 3.3.5. It is known that W_{G_v} and W_{M_v} have orders 51840 and 1920, respectively, and so that W^{M_v} has order 27. Also, it is known that $d_v = \frac{1}{2}(78 - 46) = 16$. Unlike in the classical cases, it is not easy to describe all weights of the form $w\rho_{G_v} = \rho_{G_v} + w \cdot 0$ for some $w \in W^{M_v}$, which were the weights $\kappa_v^{(s)}$ we explicitly wrote down, in terms of simple-minded operations such as permutations or changes of signs. On the other hand, since the weight space can be embedded in an ambient space of dimension only 6, we can exhaust all 27 possibilities of $w\rho_{G_v}$ (for $w \in W^{M_v}$) by direct calculation, without analyzing W_{G_v} at all. Our calculations are summarized in Tables 1 and 2 (on pages 30 and 31, respectively), which correspond to the two cases of r_v. Consequently, $\lambda_v^{(s)}$ is regular if and only if the pairings between $\lambda_v^{(s)}$ and the 27 weights $w\rho_{G_v}$ have a unique maximum at $\kappa^{(s)} := w_v^{(s)}\rho_{G_v}$ for some $w_v^{(s)} \in W^{M_v}$, in which case we define $l_v^{(s)} := l(w_v^{(s)})$ by looking up the table (with the prescribed r_v), and define $d_v^{(s)} := d_v - l_v^{(s)}$ as in (5.3). Note that one can move between the

TABLE 1. $\{w\rho_{G_v}\}_{w \in W^{M_v}}$ in the case of type E_6 (with $r_v = 1$)

$\kappa = w\rho_{G_v} = \rho_{G_v} + w \cdot 0$	$l(w)$	$w \in W^M$
$\kappa_0 = (4,3,2,1,0,4\sqrt{3})$	0	1
$\kappa_1 = (3,4,2,1,0,4\sqrt{3})$	1	$w_1 = s_1$
$\kappa_2 = (2,4,3,1,0,4\sqrt{3})$	2	$w_2 = w_1 s_2$
$\kappa_3 = (1,4,3,2,0,4\sqrt{3})$	3	$w_3 = w_2 s_3$
$\kappa_{4_I} = (0,4,3,2,-1,4\sqrt{3})$	4	$w_{4_I} = w_3 s_5$
$\kappa_{4_{II}} = (0,4,3,2,1,4\sqrt{3})$	4	$w_{4_{II}} = w_3 s_4$
$\kappa_{5_I} = (-\frac{1}{2},\frac{9}{2},\frac{7}{2},\frac{5}{2},-\frac{3}{2},\frac{7}{2}\sqrt{3})$	5	$w_{5_I} = w_{4_I} s_6$
$\kappa_{5_{II}} = (-1,4,3,2,0,4\sqrt{3})$	5	$w_{5_{II}} = w_{4_{II}} s_5$
$\kappa_{6_I} = (-\frac{3}{2},\frac{9}{2},\frac{7}{2},\frac{5}{2},-\frac{1}{2},\frac{7}{2}\sqrt{3})$	6	$w_{6_I} = w_{5_I} s_4 = w_{5_{II}} s_6$
$\kappa_{6_{II}} = (-2,4,3,1,0,4\sqrt{3})$	6	$w_{6_{II}} = w_{5_{II}} s_3$
$\kappa_{7_I} = (-\frac{5}{2},\frac{9}{2},\frac{7}{2},\frac{3}{2},-\frac{1}{2},\frac{7}{2}\sqrt{3})$	7	$w_{7_I} = w_{6_I} s_3 = w_{6_{II}} s_6$
$\kappa_{7_{II}} = (-3,4,2,1,0,4\sqrt{3})$	7	$w_{7_{II}} = w_{6_{II}} s_2$
$\kappa_{8_I} = (-3,5,4,1,0,4\sqrt{3})$	8	$w_{8_I} = w_{7_I} s_5$
$\kappa_{8_{II}} = (-\frac{7}{2},\frac{9}{2},\frac{5}{2},\frac{3}{2},-\frac{1}{2},\frac{7}{2}\sqrt{3})$	8	$w_{8_{II}} = w_{7_I} s_2 = w_{7_{II}} s_6$
$\kappa_{8_{III}} = (-4,3,2,1,0,4\sqrt{3})$	8	$w_{8_{III}} = w_{7_{II}} s_1$
$\kappa_{9_I} = (-4,5,3,1,0,4\sqrt{3})$	9	$w_{9_I} = w_{8_I} s_2 = w_{8_{II}} s_5$
$\kappa_{9_{II}} = (-\frac{9}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},-\frac{1}{2},\frac{7}{2}\sqrt{3})$	9	$w_{9_{II}} = w_{8_{II}} s_1 = w_{8_{III}} s_6$
$\kappa_{10_I} = (-\frac{9}{2},\frac{11}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{5}{2}\sqrt{3})$	10	$w_{10_I} = w_{9_I} s_3$
$\kappa_{10_{II}} = (-5,4,3,1,0,3\sqrt{3})$	10	$w_{10_{II}} = w_{9_I} s_1 = w_{9_{II}} s_5$
$\kappa_{11_I} = (-5,6,2,1,0,2\sqrt{3})$	11	$w_{11_I} = w_{10_I} s_4$
$\kappa_{11_{II}} = (-\frac{11}{2},\frac{9}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{5}{2}\sqrt{3})$	11	$w_{11_{II}} = w_{10_I} s_1 = w_{10_{II}} s_3$
$\kappa_{12_I} = (-6,5,2,1,0,2\sqrt{3})$	12	$w_{12_I} = w_{11_I} s_1$
$\kappa_{12_{II}} = (-6,4,3,2,1,2\sqrt{3})$	12	$w_{12_{II}} = w_{11_{II}} s_2$
$\kappa_{13} = (-\frac{13}{2},\frac{9}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{3}{2}\sqrt{3})$	13	$w_{13} = w_{12_I} s_2 = w_{12_{II}} s_4$
$\kappa_{14} = (-7,4,3,1,0,\sqrt{3})$	14	$w_{14} = w_{13} s_3$
$\kappa_{15} = (-\frac{15}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},-\frac{1}{2},\frac{1}{2}\sqrt{3})$	15	$w_{15} = w_{14} s_5$
$\kappa_{16} = (-8,3,2,1,0,0)$	16	$w_{16} = w_{15} s_6$

TABLE 2. $\{w\rho_{\mathrm{G}_v}\}_{w\in \mathrm{W}^{\mathrm{M}_v}}$ in the case of type E_6 (with $r_v = 6$)

$\kappa = w\rho_{\mathrm{G}_v} = \rho_{\mathrm{G}_v} + w\cdot 0$	$l(w)$	$w \in \mathrm{W}^{\mathrm{M}}$
$\kappa_0 = (4,3,2,1,0,4\sqrt{3})$	0	1
$\kappa_1 = (\frac{9}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{7}{2}\sqrt{3})$	1	$w_1 = s_6$
$\kappa_2 = (5,4,3,1,0,3\sqrt{3})$	2	$w_2 = w_1 s_5$
$\kappa_3 = (\frac{11}{2},\frac{9}{2},\frac{5}{2},\frac{3}{2},-\frac{1}{2},\frac{5}{2}\sqrt{3})$	3	$w_3 = w_2 s_3$
$\kappa_{4_{\mathrm{I}}} = (6,4,3,2,-1,2\sqrt{3})$	4	$w_{4_{\mathrm{I}}} = w_3 s_2$
$\kappa_{4_{\mathrm{II}}} = (6,5,2,1,0,2\sqrt{3})$	4	$w_{4_{\mathrm{II}}} = w_3 s_4$
$\kappa_{5_{\mathrm{I}}} = (\frac{11}{2},\frac{9}{2},\frac{7}{2},\frac{5}{2},-\frac{3}{2},\frac{3}{2}\sqrt{3})$	5	$w_{5_{\mathrm{I}}} = w_{4_{\mathrm{I}}} s_1$
$\kappa_{5_{\mathrm{II}}} = (\frac{13}{2},\frac{9}{2},\frac{5}{2},\frac{3}{2},-\frac{1}{2},\frac{3}{2}\sqrt{3})$	5	$w_{5_{\mathrm{II}}} = w_{4_{\mathrm{II}}} s_2$
$\kappa_{6_{\mathrm{I}}} = (6,5,3,2,-1,\sqrt{3})$	6	$w_{6_{\mathrm{I}}} = w_{5_{\mathrm{I}}} s_4 = w_{5_{\mathrm{II}}} s_1$
$\kappa_{6_{\mathrm{II}}} = (7,4,3,1,0,\sqrt{3})$	6	$w_{6_{\mathrm{II}}} = w_{5_{\mathrm{II}}} s_3$
$\kappa_{7_{\mathrm{I}}} = (\frac{13}{2},\frac{9}{2},\frac{7}{2},\frac{3}{2},-\frac{1}{2},\frac{1}{2}\sqrt{3})$	7	$w_{7_{\mathrm{I}}} = w_{6_{\mathrm{I}}} s_3 = w_{6_{\mathrm{II}}} s_1$
$\kappa_{7_{\mathrm{II}}} = (\frac{15}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2}\sqrt{3})$	7	$w_{7_{\mathrm{II}}} = w_{6_{\mathrm{II}}} s_5$
$\kappa_{8_{\mathrm{I}}} = (6,5,4,1,0,0)$	8	$w_{8_{\mathrm{I}}} = w_{7_{\mathrm{I}}} s_2$
$\kappa_{8_{\mathrm{II}}} = (7,4,3,2,0,0)$	8	$w_{8_{\mathrm{II}}} = w_{7_{\mathrm{I}}} s_5 = w_{7_{\mathrm{II}}} s_1$
$\kappa_{8_{\mathrm{III}}} = (8,3,2,1,0,0)$	8	$w_{8_{\mathrm{III}}} = w_{7_{\mathrm{II}}} s_6$
$\kappa_{9_{\mathrm{I}}} = (\frac{13}{2},\frac{9}{2},\frac{7}{2},\frac{3}{2},\frac{1}{2},-\frac{1}{2}\sqrt{3})$	9	$w_{9_{\mathrm{I}}} = w_{8_{\mathrm{I}}} s_5 = w_{8_{\mathrm{II}}} s_2$
$\kappa_{9_{\mathrm{II}}} = (\frac{15}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},-\frac{1}{2},-\frac{1}{2}\sqrt{3})$	9	$w_{9_{\mathrm{II}}} = w_{8_{\mathrm{II}}} s_6 = w_{8_{\mathrm{III}}} s_1$
$\kappa_{10_{\mathrm{I}}} = (6,5,3,2,1,-\sqrt{3})$	10	$w_{10_{\mathrm{I}}} = w_{9_{\mathrm{I}}} s_3$
$\kappa_{10_{\mathrm{II}}} = (7,4,3,1,0,-\sqrt{3})$	10	$w_{10_{\mathrm{II}}} = w_{9_{\mathrm{I}}} s_6 = w_{9_{\mathrm{II}}} s_2$
$\kappa_{11_{\mathrm{I}}} = (\frac{11}{2},\frac{9}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},-\frac{3}{2}\sqrt{3})$	11	$w_{11_{\mathrm{I}}} = w_{10_{\mathrm{I}}} s_4$
$\kappa_{11_{\mathrm{II}}} = (\frac{13}{2},\frac{9}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2},-\frac{3}{2}\sqrt{3})$	11	$w_{11_{\mathrm{II}}} = w_{10_{\mathrm{I}}} s_6 = w_{10_{\mathrm{II}}} s_3$
$\kappa_{12_{\mathrm{I}}} = (6,4,3,2,1,-2\sqrt{3})$	12	$w_{12_{\mathrm{I}}} = w_{11_{\mathrm{I}}} s_6$
$\kappa_{12_{\mathrm{II}}} = (6,5,2,1,0,-2\sqrt{3})$	12	$w_{12_{\mathrm{II}}} = w_{11_{\mathrm{II}}} s_5$
$\kappa_{13} = (\frac{11}{2},\frac{9}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2},-\frac{5}{2}\sqrt{3})$	13	$w_{13} = w_{12_{\mathrm{I}}} s_5 = w_{12_{\mathrm{II}}} s_4$
$\kappa_{14} = (5,4,3,1,0,-3\sqrt{3})$	14	$w_{14} = w_{13} s_3$
$\kappa_{15} = (\frac{9}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},-\frac{1}{2},-\frac{7}{2}\sqrt{3})$	15	$w_{15} = w_{14} s_2$
$\kappa_{16} = (4,3,2,1,0,-4\sqrt{3})$	16	$w_{16} = w_{15} s_1$

two cases of r_v using the reflection

$$\begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{5.29}$$

which swaps the pair of roots $\alpha_{v,1}$ and $\alpha_{v,6}$, and also the pair of roots $\alpha_{v,2}$ and $\alpha_{v,5}$, while preserving each of $\alpha_{v,3}$ and $\alpha_{v,4}$. (While the two cases are essentially the same thanks to this reflection, the actual coordinates are rather different, and hence we have still chosen to record the results in both cases. The case with $r_v = 1$ has the advantage of being more similar to the type E_7 case below, while the case with $r_v = 6$ has the advantage that the weights of M_v are easier to work with.)

Table 3. $\{w\rho_{G_v}\}_{w\in W^{M_v}}$ in the case of type E_7 (first half)

$\kappa = w\rho_{G_v} = \rho_{G_v} + w\cdot 0$	$l(w)$	$w \in W^M$
$\kappa_0 = (5,4,3,2,1,0,\frac{17}{2}\sqrt{2})$	0	1
$\kappa_1 = (4,5,3,2,1,0,\frac{17}{2}\sqrt{2})$	1	$w_1 = s_1$
$\kappa_2 = (3,5,4,2,1,0,\frac{17}{2}\sqrt{2})$	2	$w_2 = w_1s_2$
$\kappa_3 = (2,5,4,3,1,0,\frac{17}{2}\sqrt{2})$	3	$w_3 = w_2s_3$
$\kappa_4 = (1,5,4,3,2,0,\frac{17}{2}\sqrt{2})$	4	$w_4 = w_3s_4$
$\kappa_{5_{\mathrm{I}}} = (0,5,4,3,2,1,\frac{17}{2}\sqrt{2})$	5	$w_{5_{\mathrm{I}}} = w_4s_5$
$\kappa_{5_{\mathrm{II}}} = (0,5,4,3,2,-1,\frac{17}{2}\sqrt{2})$	5	$w_{5_{\mathrm{II}}} = w_4s_6$
$\kappa_{6_{\mathrm{I}}} = (-1,5,4,3,2,0,\frac{17}{2}\sqrt{2})$	6	$w_{6_{\mathrm{I}}} = w_{5_{\mathrm{I}}}s_6$
$\kappa_{6_{\mathrm{II}}} = (-\frac{1}{2},\frac{11}{2},\frac{9}{2},\frac{7}{2},\frac{5}{2},-\frac{3}{2},8\sqrt{2})$	6	$w_{6_{\mathrm{II}}} = w_{5_{\mathrm{II}}}s_7$
$\kappa_{7_{\mathrm{I}}} = (-2,5,4,3,1,0,\frac{17}{2}\sqrt{2})$	7	$w_{7_{\mathrm{I}}} = w_{6_{\mathrm{I}}}s_4$
$\kappa_{7_{\mathrm{II}}} = (-\frac{3}{2},\frac{11}{2},\frac{9}{2},\frac{7}{2},\frac{5}{2},-\frac{1}{2},8\sqrt{2})$	7	$w_{7_{\mathrm{II}}} = w_{6_{\mathrm{II}}}s_5$
$\kappa_{8_{\mathrm{I}}} = (-3,5,4,2,1,0,\frac{17}{2}\sqrt{2})$	8	$w_{8_{\mathrm{I}}} = w_{7_{\mathrm{I}}}s_3$
$\kappa_{8_{\mathrm{II}}} = (-\frac{5}{2},\frac{11}{2},\frac{9}{2},\frac{7}{2},\frac{3}{2},-\frac{1}{2},8\sqrt{2})$	8	$w_{8_{\mathrm{II}}} = w_{7_{\mathrm{II}}}s_4$
$\kappa_{9_{\mathrm{I}}} = (-4,5,3,2,1,0,\frac{17}{2}\sqrt{2})$	9	$w_{9_{\mathrm{I}}} = w_{8_{\mathrm{I}}}s_2$
$\kappa_{9_{\mathrm{II}}} = (-\frac{7}{2},\frac{11}{2},\frac{9}{2},\frac{5}{2},\frac{3}{2},-\frac{1}{2},8\sqrt{2})$	9	$w_{9_{\mathrm{II}}} = w_{8_{\mathrm{II}}}s_3$
$\kappa_{9_{\mathrm{III}}} = (-3,6,5,4,1,0,\frac{15}{2}\sqrt{2})$	9	$w_{9_{\mathrm{III}}} = w_{8_{\mathrm{II}}}s_6$
$\kappa_{10_{\mathrm{I}}} = (-5,4,3,2,1,0,\frac{17}{2}\sqrt{2})$	10	$w_{10_{\mathrm{I}}} = w_{9_{\mathrm{I}}}s_1$
$\kappa_{10_{\mathrm{II}}} = (-\frac{9}{2},\frac{11}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},-\frac{1}{2},8\sqrt{2})$	10	$w_{10_{\mathrm{II}}} = w_{9_{\mathrm{I}}}s_7 = w_{9_{\mathrm{II}}}s_2$
$\kappa_{10_{\mathrm{III}}} = (-4,6,5,3,1,0,\frac{15}{2}\sqrt{2})$	10	$w_{10_{\mathrm{III}}} = w_{9_{\mathrm{III}}}s_3$
$\kappa_{11_{\mathrm{I}}} = (-\frac{11}{2},\frac{9}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},-\frac{1}{2},8\sqrt{2})$	11	$w_{11_{\mathrm{I}}} = w_{10_{\mathrm{I}}}s_7 = w_{10_{\mathrm{II}}}s_1$
$\kappa_{11_{\mathrm{II}}} = (-5,6,4,3,1,0,\frac{15}{2}\sqrt{2})$	11	$w_{11_{\mathrm{II}}} = w_{10_{\mathrm{II}}}s_6 = w_{10_{\mathrm{III}}}s_2$
$\kappa_{11_{\mathrm{III}}} = (-\frac{9}{2},\frac{13}{2},\frac{11}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2},7\sqrt{2})$	11	$w_{11_{\mathrm{III}}} = w_{10_{\mathrm{III}}}s_4$
$\kappa_{12_{\mathrm{I}}} = (-6,5,4,3,1,0,\frac{15}{2}\sqrt{2})$	12	$w_{12_{\mathrm{I}}} = w_{11_{\mathrm{I}}}s_6 = w_{11_{\mathrm{II}}}s_1$
$\kappa_{12_{\mathrm{II}}} = (-\frac{11}{2},\frac{13}{2},\frac{9}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2},7\sqrt{2})$	12	$w_{12_{\mathrm{II}}} = w_{11_{\mathrm{II}}}s_4 = w_{11_{\mathrm{III}}}s_2$
$\kappa_{12_{\mathrm{III}}} = (-5,7,6,2,1,0,\frac{13}{2}\sqrt{2})$	12	$w_{12_{\mathrm{III}}} = w_{11_{\mathrm{III}}}s_5$
$\kappa_{13_{\mathrm{I}}} = (-\frac{13}{2},\frac{11}{2},\frac{9}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2},7\sqrt{2})$	13	$w_{13_{\mathrm{I}}} = w_{12_{\mathrm{I}}}s_4 = w_{12_{\mathrm{II}}}s_1$
$\kappa_{13_{\mathrm{II}}} = (-6,7,4,3,2,1,\frac{13}{2}\sqrt{2})$	13	$w_{13_{\mathrm{II}}} = w_{12_{\mathrm{II}}}s_3$
$\kappa_{13_{\mathrm{III}}} = (-6,7,5,2,1,0,\frac{13}{2}\sqrt{2})$	13	$w_{13_{\mathrm{III}}} = w_{12_{\mathrm{II}}}s_5 = w_{12_{\mathrm{III}}}s_2$

5.2.6. *Type* E_7. Suppose we are in the context of Section 3.3.6. It is known that W_{G_v} and W_{M_v} have orders 2903040 and 51840, respectively, and so that W^{M_v} has order 56. Also, it is known that $d_v = \frac{1}{2}(133-79) = 27$. Again, since the weight space can be embedded in an ambient space of dimension only 7, we can exhaust all 56 possibilities of $w\rho_{G_v}$ (for $w \in W^{M_v}$) by direct calculation, without analyzing W_{G_v} at all. Our calculations are summarized in Tables 3 and 4 (on pages 32 and 33, respectively). Consequently, $\lambda_v^{(s)}$ is regular if and only if the pairings between $\lambda_v^{(s)}$ and the 56 weights $w\rho_{G_v}$ have a unique maximum at $\kappa^{(s)} := w_v^{(s)}\rho_{G_v}$ for some $w_v^{(s)} \in W^{M_v}$, in which case we define $l_v^{(s)} := l(w_v^{(s)})$ by looking up the tables, and define $d_v^{(s)} := d_v - l_v^{(s)}$ as in (5.3).

5.3. **Examples.** In this subsection, for simplicity, we shall drop the index v when there is only one $\mathbb{C}$-simple factor.

5.3.1. *Type* A. Let us continue with the setting of Section 5.2.1.

TABLE 4. $\{w\rho_{\mathrm{G}_v}\}_{w\in \mathrm{W}^{\mathrm{M}_v}}$ in the case of type E_7 (second half)

$\kappa = w\rho_{\mathrm{G}_v} = \rho_{\mathrm{G}_v} + w\cdot 0$	$l(w)$	$w \in \mathrm{W}^{\mathrm{M}}$
$\kappa_{14_{\mathrm{I}}} = (-7,6,4,3,2,1,\frac{13}{2}\sqrt{2})$	14	$w_{14_{\mathrm{I}}} = w_{13_{\mathrm{I}}}s_3 = w_{13_{\mathrm{II}}}s_1$
$\kappa_{14_{\mathrm{II}}} = (-\frac{13}{2},\frac{15}{2},\frac{9}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2},6\sqrt{2})$	14	$w_{14_{\mathrm{II}}} = w_{13_{\mathrm{II}}}s_5 = w_{13_{\mathrm{III}}}s_3$
$\kappa_{14_{\mathrm{III}}} = (-7,6,5,2,1,0,\frac{13}{2}\sqrt{2})$	14	$w_{14_{\mathrm{III}}} = w_{13_{\mathrm{III}}}s_1$
$\kappa_{15_{\mathrm{I}}} = (-\frac{15}{2},\frac{11}{2},\frac{9}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},6\sqrt{2})$	15	$w_{15_{\mathrm{I}}} = w_{14_{\mathrm{I}}}s_2$
$\kappa_{15_{\mathrm{II}}} = (-7,8,4,3,1,0,\frac{11}{2}\sqrt{2})$	15	$w_{15_{\mathrm{II}}} = w_{14_{\mathrm{II}}}s_4$
$\kappa_{15_{\mathrm{III}}} = (-\frac{15}{2},\frac{13}{2},\frac{9}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2},6\sqrt{2})$	15	$w_{15_{\mathrm{III}}} = w_{14_{\mathrm{II}}}s_1 = w_{14_{\mathrm{III}}}s_3$
$\kappa_{16_{\mathrm{I}}} = (-8,6,5,3,2,1,\frac{11}{2}\sqrt{2})$	16	$w_{16_{\mathrm{I}}} = w_{15_{\mathrm{I}}}s_5 = w_{15_{\mathrm{III}}}s_2$
$\kappa_{16_{\mathrm{II}}} = (-8,7,4,3,1,0,\frac{11}{2}\sqrt{2})$	16	$w_{16_{\mathrm{II}}} = w_{15_{\mathrm{II}}}s_1$
$\kappa_{16_{\mathrm{III}}} = (-\frac{15}{2},\frac{17}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},-\frac{1}{2},5\sqrt{2})$	16	$w_{16_{\mathrm{III}}} = w_{15_{\mathrm{II}}}s_6$
$\kappa_{17_{\mathrm{I}}} = (-\frac{17}{2},\frac{13}{2},\frac{9}{2},\frac{7}{2},\frac{3}{2},\frac{1}{2},5\sqrt{2})$	17	$w_{17_{\mathrm{I}}} = w_{16_{\mathrm{I}}}s_4$
$\kappa_{17_{\mathrm{II}}} = (-\frac{17}{2},\frac{15}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},-\frac{1}{2},5\sqrt{2})$	17	$w_{17_{\mathrm{II}}} = w_{16_{\mathrm{II}}}s_6$
$\kappa_{17_{\mathrm{III}}} = (-8,9,3,2,1,0,\frac{9}{2}\sqrt{2})$	17	$w_{17_{\mathrm{III}}} = w_{16_{\mathrm{III}}}s_7$
$\kappa_{18_{\mathrm{I}}} = (-9,6,5,4,1,0,\frac{9}{2}\sqrt{2})$	18	$w_{18_{\mathrm{I}}} = w_{17_{\mathrm{I}}}s_3$
$\kappa_{18_{\mathrm{II}}} = (-9,7,4,3,2,0,\frac{9}{2}\sqrt{2})$	18	$w_{18_{\mathrm{II}}} = w_{17_{\mathrm{II}}}s_2$
$\kappa_{18_{\mathrm{III}}} = (-9,8,3,2,1,0,\frac{9}{2}\sqrt{2})$	18	$w_{18_{\mathrm{III}}} = w_{17_{\mathrm{III}}}s_1$
$\kappa_{19_{\mathrm{I}}} = (-\frac{19}{2},\frac{13}{2},\frac{9}{2},\frac{7}{2},\frac{3}{2},-\frac{1}{2},4\sqrt{2})$	19	$w_{19_{\mathrm{I}}} = w_{18_{\mathrm{I}}}s_6 = w_{18_{\mathrm{II}}}s_3$
$\kappa_{19_{\mathrm{II}}} = (-\frac{19}{2},\frac{15}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2},4\sqrt{2})$	19	$w_{19_{\mathrm{II}}} = w_{18_{\mathrm{II}}}s_7 = w_{18_{\mathrm{III}}}s_2$
$\kappa_{20_{\mathrm{I}}} = (-10,7,4,3,1,0,\frac{7}{2}\sqrt{2})$	20	$w_{20_{\mathrm{I}}} = w_{19_{\mathrm{I}}}s_7 = w_{19_{\mathrm{II}}}s_3$
$\kappa_{20_{\mathrm{II}}} = (-10,6,5,3,2,-1,\frac{7}{2}\sqrt{2})$	20	$w_{20_{\mathrm{II}}} = w_{19_{\mathrm{I}}}s_4$
$\kappa_{21_{\mathrm{I}}} = (-\frac{21}{2},\frac{13}{2},\frac{9}{2},\frac{5}{2},\frac{3}{2},-\frac{1}{2},3\sqrt{2})$	21	$w_{21_{\mathrm{I}}} = w_{20_{\mathrm{I}}}s_4$
$\kappa_{21_{\mathrm{II}}} = (-\frac{21}{2},\frac{11}{2},\frac{9}{2},\frac{7}{2},\frac{5}{2},-\frac{3}{2},3\sqrt{2})$	21	$w_{21_{\mathrm{II}}} = w_{20_{\mathrm{II}}}s_5$
$\kappa_{22_{\mathrm{I}}} = (-11,6,5,2,1,0,\frac{5}{2}\sqrt{2})$	22	$w_{22_{\mathrm{I}}} = w_{21_{\mathrm{I}}}s_6$
$\kappa_{22_{\mathrm{II}}} = (-11,6,4,3,2,-1,\frac{5}{2}\sqrt{2})$	22	$w_{22_{\mathrm{II}}} = w_{21_{\mathrm{II}}}s_7$
$\kappa_{23_{\mathrm{I}}} = (-\frac{23}{2},\frac{11}{2},\frac{9}{2},\frac{5}{2},\frac{3}{2},-\frac{1}{2},2\sqrt{2})$	23	$w_{23_{\mathrm{I}}} = w_{22_{\mathrm{I}}}s_5 = w_{22_{\mathrm{II}}}s_6$
$\kappa_{24_{\mathrm{I}}} = (-12,5,4,3,1,0,\frac{3}{2}\sqrt{2})$	24	$w_{24_{\mathrm{I}}} = w_{23_{\mathrm{I}}}s_4$
$\kappa_{25_{\mathrm{I}}} = (-\frac{25}{2},\frac{9}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2},\sqrt{2})$	25	$w_{25_{\mathrm{I}}} = w_{24_{\mathrm{I}}}s_3$
$\kappa_{26_{\mathrm{I}}} = (-13,4,3,2,1,0,\frac{1}{2}\sqrt{2})$	26	$w_{26_{\mathrm{I}}} = w_{25_{\mathrm{I}}}s_2$
$\kappa_{27_{\mathrm{I}}} = (-13,4,3,2,1,0,-\frac{1}{2}\sqrt{2})$	27	$w_{27_{\mathrm{I}}} = w_{26_{\mathrm{I}}}s_1$

Example 5.30 ($\mathbb{C}$-simple type A_2 with $r=2$). Suppose $\nu = (3,2;-1)$, so that

$$\lambda^{(0)} = \lambda + \rho_{\mathrm{G}} = (-2,-3,1) + (1,0,-1) = (-1,-3,0).$$

Let us represent the fundamental weight by $\varpi_0 = (1,1,0)$. Then $[d^-,d^+] = [0,1]$, which is the same interval we have seen in Example 4.19 (for the point there with coordinates $(4,3) = (3-(-1), 2-(-1))$), by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d^{(s)}$
1	$(0,-2,0)$	not regular	
2	$(1,-1,0)$	$(1,-1,0)$	$1+0=1$
-1	$(-2,-4,0)$	$(0,-1,1)$	1 - 1 = 0

Example 5.31 ($\mathbb{C}$-simple type A_3 with $r=2$). Suppose $\nu = (3,1;2,-2)$, so that

$$\lambda^{(0)} = \lambda + \rho_{\mathrm{G}} = (-1,-3,2,-2) + (\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2}) = (\tfrac{1}{2},-\tfrac{5}{2},\tfrac{3}{2},-\tfrac{7}{2}).$$

Let us represent the fundamental weight by $\varpi_0 = (1,1,0,0)$. Then $[d^-, d^+] = [1,3]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d^{(s)}$
1	$(\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{7}{2})$	not regular	
2	$(\frac{5}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{7}{2})$	$(\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{3}{2})$	$2+1=3$
-1	$(-\frac{1}{2}, -\frac{7}{2}, \frac{3}{2}, -\frac{7}{2})$	not regular	
-2	$(-\frac{3}{2}, -\frac{9}{2}, \frac{3}{2}, -\frac{7}{2})$	$(\frac{1}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{1}{2})$	$2-1=1$

Example 5.32 (two $\mathbb{C}$-simple factors of type A_6 in the same $\mathbb{Q}$-simple factor). Suppose the group is $\mathbb{Q}$-simple but has two $\mathbb{C}$-simple factors of A_6, which we denote by $\upsilon = 1$ and 2, respectively. Suppose $r_1 = 3$ and $r_2 = 2$, so that $d = d_1 + d_2 = 12 + 10 = 22$. Suppose $(\nu_1, \nu_2) = ((6,2,-5;2,0,0,-4),(2,-5;8,4,2,0,0))$, so that

$$\begin{aligned} \lambda^{(0)} = \lambda + (\rho_{G_1}, \rho_{G_2}) &= ((5,-2,-6,4,0,0,-2),(5,-2,0,0,-2,-4,-8)) \\ &\quad + ((3,2,1,0,-1,-2,-3),(3,2,1,0,-1,-2,-3)) \\ &= ((8,0,-5,4,-1,-2,-5),(8,0,1,0,-3,-6,-11)). \end{aligned}$$

Let us represent the fundamental weights by $\varpi_{1,0} = (1,1,1;0,0,0,0)$ and $\varpi_{2,0} = (1,1,0,0,0,0,0)$. Then $[d^-, d^+] = [3,18]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d^{(s)}$
1	((9,1,−4,4,−1,−2,−5), (9,1,1,0,−3,−6,−11))	not regular (2nd factor)	
2	((10,2,−3,4,−1,−2,−5), (10,2,1,0,−3,−6,−11))	((3,1,−2,2,0,−1,−3), (3,2,1,0,−1,−2,−3))	$11+7=18$
-1	((7,−1,−6,4,−1,−2,−5), (7,−1,1,0,−3,−6,−11))	not regular (1st factor)	
-2	((6,−2,−7,4,−1,−2,−5), (6,−2,1,0,−3,−6,−11))	not regular (1st factor)	
-3	((5,−3,−8,4,−1,−2,−5), (5,−3,1,0,−3,−6,−11))	not regular (2nd factor)	
-4	((4,−4,−9,4,−1,−2,−5), (4,−4,1,0,−3,−6,−11))	not regular (1st factor)	
-5	((3,−5,−10,4,−1,−2,−5), (3,−5,1,0,−3,−6,−11))	not regular (2nd factor)	
-6	((2,−6,−11,4,−1,−2,−5), (2,−6,1,0,−3,−6,−11))	not regular (2nd factor)	
-7	((1,−7,−12,4,−1,−2,−5), (1,−7,1,0,−3,−6,−11))	not regular (2nd factor)	
-8	((0,−8,−12,4,−1,−2,−5), (0,−8,1,0,−3,−6,−11))	not regular (2nd factor)	
-9	((−1,−9,−13,4,−1,−2,−5), (−1,−9,1,0,−3,−6,−11))	not regular (1st factor)	
-10	((−2,−10,−14,4,−1,−2,−5), (−2,−10,1,0,−3,−6,−11))	not regular (1st factor)	
-11	((−3,−11,−15,4,−1,−2,−5), (−3,−11,1,0,−3,−6,−11))	not regular (2nd factor)	
-12	((−4,−12,−16,4,−1,−2,−5), (−4,−12,1,0,−3,−6,−11))	((0,−2,−3,3,2,1,−1), (0,−3,3,2,1,−1−2))	$11-8=3$

If the two $\mathbb{C}$-simple factors were not in the same $\mathbb{Q}$-simple factor, then $\nu_1 = (6,2,-5;2,0,0,-4)$ and $\nu_2 = (2,-5;8,4,2,0,0)$ would have defined the intervals $[5,8]$ and $[8,10]$ on their respective factors (by similar calculations), whose end points sum up to those of $[13,18]$, which is much narrower than $[3,18]$.

5.3.2. *Type* B. Let us continue with the setting of Section 5.2.2.

Example 5.33 ($\mathbb{C}$-simple type B_2, with $r = 1$). Suppose $\nu = (-2;3)$, so that

$$\lambda^{(0)} = \lambda + \rho_G = (2,3) + (\tfrac{3}{2}, \tfrac{1}{2}) = (\tfrac{7}{2}, \tfrac{7}{2}).$$

Then $[d^-, d^+] = [2, 3]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d^{(s)}$
1	$(\frac{9}{2}, \frac{7}{2})$	$(\frac{3}{2}, \frac{1}{2})$	$\frac{3}{2} + \frac{3}{2} = 3$
-1	$(\frac{5}{2}, \frac{7}{2})$	$(\frac{1}{2}, \frac{3}{2})$	$\frac{3}{2} + \frac{1}{2} = 2$

Example 5.34 ($\mathbb{C}$-simple type B_3, with $r = 1$). Suppose $\nu = (6; 3, 2)$, so that

$$\lambda^{(0)} = \lambda + \rho_{\mathrm{G}} = (-6, 3, 2) + (\tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}) = (-\tfrac{7}{2}, \tfrac{9}{2}, \tfrac{5}{2}).$$

Then $[d^-, d^+] = [0, 2]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d^{(s)}$
1	$(-\frac{5}{2}, \frac{9}{2}, \frac{5}{2})$	not regular	
2	$(-\frac{3}{2}, \frac{9}{2}, \frac{5}{2})$	$(-\frac{1}{2}, \frac{5}{2}, \frac{3}{2})$	$\frac{5}{2} + \frac{-1}{2} = 2$
-1	$(-\frac{9}{2}, \frac{9}{2}, \frac{5}{2})$	not regular	
-2	$(-\frac{11}{2}, \frac{9}{2}, \frac{5}{2})$	$(-\frac{5}{2}, \frac{3}{2}, \frac{1}{2})$	$\frac{5}{2} - \frac{5}{2} = 0$

Example 5.35 (mixture of compact and noncompact $\mathbb{C}$-simple factors of type B_4 in a $\mathbb{Q}$-simple factor). Suppose the group is $\mathbb{Q}$-simple but has two $\mathbb{C}$-simple factors of B_4, which we denote by $\upsilon = 1$ and 2, respectively. Suppose that r_1 does not exist (i.e., $\mathrm{M}_1 = \mathrm{G}_1$), and that $r_2 = 1$, so that $d = d_1 + d_2 = 0 + 7 = 7$. Suppose $\nu = (\nu_1, \nu_2) = ((4, 2, 1, 1), (-2; 3, 2, 1))$, so that

$$\begin{aligned} \lambda^{(0)} = \lambda + (\rho_{\mathrm{G}_1}, \rho_{\mathrm{G}_2}) &= ((4, 2, 1, 1), (2, 3, 2, 1)) + ((\tfrac{7}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}), (\tfrac{7}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2})) \\ &= ((\tfrac{15}{2}, \tfrac{9}{2}, \tfrac{5}{2}, \tfrac{3}{2}), (\tfrac{11}{2}, \tfrac{11}{2}, \tfrac{7}{2}, \tfrac{3}{2})). \end{aligned}$$

Note that $\lambda^{(0)}$ is not regular, and hence ν is not cohomological in the sense of Definition 2.6. Then $[d^-, d^+] = [6, 7]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d^{(s)}$
1	$((\frac{15}{2}, \frac{9}{2}, \frac{5}{2}, \frac{3}{2}), (\frac{13}{2}, \frac{11}{2}, \frac{7}{2}, \frac{3}{2}))$	$((\frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}), (\frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}))$	$\frac{7}{2} + \frac{7}{2} = 7$
-1	$((\frac{15}{2}, \frac{9}{2}, \frac{5}{2}, \frac{3}{2}), (\frac{9}{2}, \frac{11}{2}, \frac{7}{2}, \frac{3}{2}))$	$((\frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}), (\frac{5}{2}, \frac{7}{2}, \frac{3}{2}, \frac{1}{2}))$	$\frac{7}{2} + \frac{5}{2} = 6$

Note that the first factor (for which r_1 does not exist) contributes trivially to the calculation of cohomological degrees. Such factors can be harmless omitted even in general (cf. the beginning of Section 5.2). Also note that the associated locally symmetric variety X is compact in this case. Even so, the coherent cohomology of automorphic bundles with noncohomological weights (namely, not contributing to the de Rham cohomology) might still be nonzero, although it is a subtle question (which is unsolved in general) whether this is indeed the case.

5.3.3. *Type* C. Let us continue with the setting of Section 5.2.3.

Example 5.36 ($\mathbb{C}$-simple type C_2, with $r = 2$). Suppose $\nu = (4, 1)$, so that

$$\lambda^{(0)} = \lambda + \rho_{\mathrm{G}} = (-1, -4) + (2, 1) = (1, -3).$$

Then $[d^-, d^+] = [0, 2]$, which is the same interval we have seen in Example 4.17, by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d^{(s)}$
1	$(2, -2)$	not regular	
2	$(3, -1)$	$(2, -1)$	$\frac{1}{2}(3 + 1) = 2$
-1	$(0, -4)$	not regular	
-2	$(-1, -5)$	$(-1, -2)$	$\frac{1}{2}(3 - 3) = 0$

Example 5.37 ($\mathbb{C}$-simple type C_3, with $r = 3$). Suppose $\nu = (3, 1, -7)$, so that

$$\lambda^{(0)} = \lambda + \rho_{\mathrm{G}} = (7, -1, -3) + (3, 2, 1) = (10, 1, -2).$$

Then $[d^-, d^+] = [3, 5]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d^{(s)}$
1	$(11, 2, -1)$	$(3, 2, -1)$	$\frac{1}{2}(6+4) = 5$
-1	$(9, 0, -3)$	not regular	
-2	$(8, -1, -4)$	$(3, -1, -2)$	$\frac{1}{2}(6+0) = 3$

Example 5.38 (restriction of scalar of type C_2). Suppose the group is $\mathbb{Q}$-simple but has two $\mathbb{C}$-simple factors of C_2, which we denote by $\upsilon = 1$ and 2, respectively, such that $r_1 = r_2 = 2$. Suppose $\nu = (\nu_1, \nu_2) = ((5, 1), (-1, -2))$, so that

$$\lambda^{(0)} = \lambda + (\rho_{\mathrm{G}_1}, \rho_{\mathrm{G}_2}) = ((-1, -5), (2, 1)) + ((2, 1), (2, 1)) = ((1, -4), (4, 2)).$$

Then $[d^-, d^+] = [0, 4]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d^{(s)}$
1	$((2, -3), (5, 3))$	$((1, -2), (2, 1))$	$\frac{1}{2}(6+2) = 4$
-1	$((0, -5), (3, 1))$	not regular (1st factor)	
-2	$((-1, -6), (2, 0))$	not regular (2nd factor)	
-3	$((-2, -7), (1, -1))$	not regular (2nd factor)	
-4	$((-3, -8), (0, -2))$	not regular (2nd factor)	
-5	$((-4, -9), (-1, -3))$	$((-1, -2), (-1, -2))$	$\frac{1}{2}(6-6) = 0$

If the two $\mathbb{C}$-simple factors were not in the same $\mathbb{Q}$-simple factor, then $\nu_1 = (5, 1)$ and $\nu_2 = (-1, -2)$ would have defined the intervals $[0, 1]$ and $[3, 3]$ on their respective factors, whose end points sum up to those of $[3, 4]$, which is much narrower than $[0, 4]$.

5.3.4. *Type* D. Let us continue with the setting of Section 5.2.4.

Example 5.39 ($\mathbb{C}$-simple type D_4 with $r = 1$; i.e., type $\mathrm{D}_4^{\mathbb{R}}$). Suppose $\nu = (1; 2, 2, 0)$, so that

$$\lambda^{(0)} = \lambda + \rho_{\mathrm{G}} = (-1, 2, 2, 0) + (3, 2, 1, 0) = (2, 4, 3, 0).$$

Then $[d^-, d^+] = [4, 6]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d^{(s)}$
1	$(3, 4, 3, 0)$	not regular	
2	$(4, 4, 3, 0)$	not regular	
3	$(5, 4, 3, 0)$	$(3, 2, 1, 0)$	$3 + 3 = 6$
-1	$(1, 4, 3, 0)$	$(1, 3, 2, 0)$	$3 + 1 = 4$

Example 5.40 ($\mathbb{C}$-simple type D_4 with $r = 4$; i.e., type $\mathrm{D}_4^{\mathbb{H}}$). Suppose $\nu = (9, 5, -2, -2)$, so that

$$\lambda^{(0)} = \lambda + \rho_{\mathrm{G}} = (2, 2, -5, -9) + (3, 2, 1, 0) = (5, 4, -4, -9).$$

Then $[d^-, d^+] = [1, 3]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d^{(s)}$
1	$(\frac{11}{2}, \frac{9}{2}, -\frac{7}{2}, -\frac{17}{2})$	$(2, 1, 0, -3)$	$3 + 0 = 3$
-1	$(\frac{9}{2}, \frac{7}{2}, -\frac{9}{2}, -\frac{19}{2})$	not regular	
-2	$(4, 3, -5, -10)$	$(1, 0, -2, -3)$	$3 - 2 = 1$

Example 5.41 (mixture of the two types in a $\mathbb{Q}$-simple factor). Suppose the group is $\mathbb{Q}$-simple but has two $\mathbb{C}$-simple factors of D_4, one being as in Example 5.39, the other being as in Example 5.40, which we denote by $\upsilon = 1$ and 2, respectively. Suppose $\nu = (\nu_1, \nu_2) = ((1; 2, 2, 0), (9, 5, -2, -2))$ (whose factors are exactly the ones we have seen). Then $\lambda^{(0)} = ((2, 4, 3, 0), (5, 4, -4, -9))$, and $[d^-, d^+] = [3, 9]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d^{(s)}$
1	$((3,4,3,0),(\frac{11}{2},\frac{9}{2},-\frac{7}{2},-\frac{17}{2}))$	not regular (1st factor)	
2	$((4,4,3,0),(6,5,-3,-8))$	not regular (1st factor)	
3	$((5,4,3,0),(\frac{13}{2},\frac{11}{2},-\frac{5}{2},-\frac{15}{2}))$	$((3,2,1,0),(2,1,0,-3))$	$6+3=9$
-1	$((1,4,3,0),(\frac{9}{2},\frac{7}{2},-\frac{9}{2},-\frac{19}{2}))$	not regular (2nd factor)	
-2	$((0,4,3,0),(4,3,-5,-10))$	not regular (1st factor)	
-3	$((-1,4,3,0),(\frac{7}{2},\frac{5}{2},-\frac{11}{2},-\frac{17}{2}))$	$((-1,3,2,0),(1,0,-2,-3))$	$6-3=3$

If the two $\mathbb{C}$-simple factors were not in the same $\mathbb{Q}$-simple factor, then $\nu_1 = (1; 2, 2, 0)$ and $\nu_2 = (9, 5, -2, -2)$ would have defined the intervals $[4, 6]$ and $[1, 3]$ on their respective factors, whose end points sum up to those of $[5, 9]$, which is narrower than $[3, 9]$.

5.3.5. *Type* E_6. Let us continue with the setting of Section 5.2.5.

Example 5.42 ($\mathbb{C}$-simple type E_6 with $r = 1$, cohomological weight). Suppose $\nu = (4; 4, 3, 1, 0, 4\sqrt{3})$, so that

$$\lambda^{(0)} = \lambda + \rho_G = (-8, 4, 3, 1, 0, 0) + (4, 3, 2, 1, 0, 4\sqrt{3}) = (-4, 7, 5, 2, 0, 4\sqrt{3}).$$

Note that $\lambda^{(0)}$ is regular, which pairs maximally with κ_{8_I} in Table 1 (on page 30), and hence ν is cohomological in the sense of Definition 2.6, with $w(\nu) = w_{8_I}$ and $\mu(\nu) = w_{8_I}^{-1}(\lambda^{(0)}) - \rho_G = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}\sqrt{3})$. Then $[d^-, d^+] = [3, 13]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d - l^{(s)} = d^{(s)}$
1	$(-3,7,5,2,0,\frac{13}{3}\sqrt{3})$	not regular	
2	$(-2,7,5,2,0,\frac{14}{3}\sqrt{3})$	not regular	
3	$(-1,7,5,2,0,5\sqrt{3})$	not regular	
4	$(-0,7,5,2,0,\frac{16}{3}\sqrt{3})$	not regular	
5	$(1,7,5,2,0,\frac{17}{3}\sqrt{3})$	κ_3 in Table 1	$16-3=13$
-1	$(-5,7,5,2,0,\frac{11}{3}\sqrt{3})$	not regular	
-2	$(-6,7,5,2,0,\frac{10}{3}\sqrt{3})$	not regular	
-3	$(-7,7,5,2,0,3\sqrt{3})$	not regular	
-4	$(-8,7,5,2,0,\frac{8}{3}\sqrt{3})$	not regular	
-5	$(-9,7,5,2,0,\frac{7}{3}\sqrt{3})$	κ_{13} in Table 1	$16-13=3$

(Since $d^- \neq d^+$, by Theorem 4.10, or rather by its proof, $\mu(\nu)$ cannot be regular.)

Example 5.43 ($\mathbb{C}$-simple type E_6 with $r = 6$, cohomological weight). Suppose $\nu = (11, 8, 3, 2, -1; 9\sqrt{3})$, so that

$$\lambda^{(0)} = \lambda + \rho_G = (11, 8, 3, 2, 1, -9\sqrt{3}) + (4, 3, 2, 1, 0, 4\sqrt{3}) = (15, 11, 5, 3, 1, -5\sqrt{3}).$$

Note that $\lambda^{(0)}$ is regular, which pairs maximally with $\kappa_{12_{II}}$ in Table 2 (on page 31), and hence ν is cohomological in the sense of Definition 2.6, with $w(\nu) = w_{12_{II}}$ and

$\mu(\nu) = w_{12_{\mathrm{II}}}^{-1}(\lambda^{(0)}) - \rho_{\mathrm{G}} = (5,4,3,0,0,6\sqrt{3})$. Then $[d^-, d^+] = [3,5]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d - l^{(s)} = d^{(s)}$
1	$(15,11,5,3,1,-\frac{13}{3}\sqrt{3})$	not regular	
2	$(15,11,5,3,1,-\frac{11}{3}\sqrt{3})$	$\kappa_{11_{\mathrm{II}}}$ in Table 2	$16-11=5$
-1	$(15,11,5,3,1,-\frac{17}{3}\sqrt{3})$	not regular	
-2	$(15,11,5,3,1,-\frac{19}{3}\sqrt{3})$	κ_{13} in Table 2	$16-13=3$

(Since $d^- \neq d^+$, by Theorem 4.10, or rather by its proof, $\mu(\nu)$ cannot be regular.)

Example 5.44 ($\mathbb{C}$-simple type E_6 with $r = 6$, noncohomological weight). Suppose $\nu = (3,1,1,0,0;\sqrt{3})$, so that

$$\lambda^{(0)} = \lambda + \rho_{\mathrm{G}} = (3,1,1,0,0,-\sqrt{3}) + (4,3,2,1,0,4\sqrt{3}) = (7,4,3,1,0,3\sqrt{3}).$$

Note that $\lambda^{(0)}$ is not regular because there is no unique maximal pairing between it and the weights in Table 2, and hence ν is not cohomological in the sense of Definition 2.6. Then $[d^-, d^+] = [10,14]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d - l^{(s)} = d^{(s)}$
1	$(7,4,3,1,0,\frac{11}{3}\sqrt{3})$	κ_2 in Table 2	$16-2=14$
-1	$(7,4,3,1,0,\frac{7}{3}\sqrt{3})$	not regular	
-2	$(7,4,3,1,0,\frac{5}{3}\sqrt{3})$	not regular	
-3	$(7,4,3,1,0,\sqrt{3})$	$\kappa_{6_{\mathrm{II}}}$ in Table 2	$16-6=10$

Example 5.45 (mixture of the two types in a $\mathbb{Q}$-simple factor). Suppose the group is $\mathbb{Q}$-simple but has two $\mathbb{C}$-simple factors of E_6, which we denote by $\upsilon = 1$ and 2, respectively, such that $r_1 = 1$ and $r_2 = 6$. Suppose $\nu = (\nu_1, \nu_2) = ((0;3,2,1,0,8\sqrt{3}),(5,4,2,1,0;10\sqrt{3}))$, so that

$$\begin{aligned}\lambda^{(0)} = \lambda + (\rho_{\mathrm{G}_1}, \rho_{\mathrm{G}_2}) &= ((-12,3,2,1,0,4\sqrt{3}),(5,4,2,1,0,-10\sqrt{3}))\\ &\quad + ((4,3,2,1,0,4\sqrt{3}),(4,3,2,1,0,4\sqrt{3}))\\ &= ((-8,6,4,2,0,8\sqrt{3}),(9,7,4,2,0,-6\sqrt{3})).\end{aligned}$$

Note that $\lambda^{(0)}$ is not regular because of its second factor, and hence ν is not cohomological in the sense of Definition 2.6. (Nevertheless, the first factor is regular, which pairs maximally with $\kappa_{8_{\mathrm{III}}}$ in Table 1.) Then $[d^-, d^+] = [9,10]$ by calculations summarized as follows (where the two factors of each weight in the column of $\kappa^{(s)}$ can be found in Tables 1 and 2, respectively, on pages 30 and 31):

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d - l^{(s)} = d^{(s)}$
1	$((-7,6,4,2,0,\frac{25}{3}\sqrt{3}),(9,7,4,2,0,-\frac{16}{3}\sqrt{3}))$	$(\kappa_{8_{\mathrm{III}}}, \kappa_{14})$	$32-22=10$
-1	$((-9,6,4,2,0,\frac{23}{3}\sqrt{3}),(9,7,4,2,0,-\frac{20}{3}\sqrt{3}))$	$(\kappa_{8_{\mathrm{III}}}, \kappa_{15})$	$32-23=9$

If the two $\mathbb{C}$-simple factors were not in the same $\mathbb{Q}$-simple factor, then $\nu_1 = (0;3,2,1,0,8\sqrt{3})$ and $\nu_2 = (5,4,2,1,0;10\sqrt{3})$ would have defined the intervals $[8,8]$ and $[1,2]$ on their respective factors, whose end points sum up to the same interval $[9,10]$. (This is certainly not always true, as we have seen in Examples 5.31, 5.38, and 5.41. See also Example 5.49 below.)

5.3.6. *Type* E_7. Let us continue with the setting of Section 5.2.6.

Example 5.46 ($\mathbb{C}$-simple type E_7 with $r = 1$, cohomological weight). Suppose $\nu = (10, 10, 9, 7, 4, 0, 26\sqrt{2})$, so that

$$\begin{aligned}\lambda^{(0)} = \lambda + \rho_{\mathrm{G}} &= (-36, 16, 10, 6, 3, -1, 0) + (5, 4, 3, 2, 1, 0, \tfrac{17}{2}\sqrt{2}) \\ &= (-31, 20, 13, 8, 4, -1, \tfrac{17}{2}\sqrt{2}).\end{aligned}$$

Note that $\lambda^{(0)}$ is regular, which pairs maximally with

$$\kappa_{21_{\mathrm{I}}} = (-\tfrac{21}{2}, \tfrac{13}{2}, \tfrac{9}{2}, \tfrac{5}{2}, \tfrac{3}{2}, -\tfrac{1}{2}, 3\sqrt{2})$$

in Table 4 (on page 33), and hence ν is cohomological in the sense of Definition 2.6, with $w(\nu) = w_{21_{\mathrm{I}}}$ and $\mu(\nu) = w_{21_{\mathrm{I}}}^{-1}(\lambda^{(0)}) - \rho_{\mathrm{G}} = (9, 8, 6, 3, 2, 0, 17\sqrt{2})$. Then we have $[d^-, d^+] = [6, 6]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d - l^{(s)} = d^{(s)}$
1	$(-30, 20, 13, 8, 4, -1, 9\sqrt{2})$	$\kappa_{21_{\mathrm{I}}}$ in Table 4	$27 - 21 = 6$
-1	$(-32, 20, 13, 8, 4, -1, 8\sqrt{2})$	$\kappa_{21_{\mathrm{I}}}$ in Table 4	$27 - 21 = 6$

Indeed, this concentration in one degree follows more directly from the regularity of $\mu(\nu)$ (as a weight in $\mathrm{X}^+_{\mathrm{G}_\upsilon}$, which can be checked more easily by pairings with the simple positive roots $\alpha_{\upsilon,1}, \ldots, \alpha_{\upsilon,7}$), and from Theorem 4.10, without having to compute $\lambda^{(1)}$ and $\lambda^{(-1)}$ at all.

Example 5.47 ($\mathbb{C}$-simple type E_7 with $r = 1$, cohomological weight). Suppose $\nu = (-14, 8, 3, 2, 1, 0, \sqrt{2})$, so that

$$\begin{aligned}\lambda^{(0)} = \lambda + \rho_{\mathrm{G}} &= (2, 4, 3, 2, 1, 0, 11\sqrt{2}) + (5, 4, 3, 2, 1, 0, \tfrac{17}{2}\sqrt{2}) \\ &= (7, 8, 6, 4, 2, 0, \tfrac{39}{2}\sqrt{2}).\end{aligned}$$

Note that $\lambda^{(0)}$ is regular, which pairs maximally with κ_1 in Table 3 (on page 32), and hence ν is cohomological in the sense of Definition 2.6, with $w(\nu) = w_1$ and $\mu(\nu) = w_1^{-1}(\lambda^{(0)}) - \rho_{\mathrm{G}} = (3, 3, 3, 2, 1, 0, 11\sqrt{2})$. Then $[d^-, d^+] = [25, 27]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d - l^{(s)} = d^{(s)}$
1	$(8, 8, 6, 4, 2, 0, 20\sqrt{2})$	not regular	
2	$(9, 8, 6, 4, 2, 0, \tfrac{41}{2}\sqrt{2})$	κ_0 in Table 3	$27 - 0 = 27$
-1	$(6, 8, 6, 4, 2, 0, 19\sqrt{2})$	not regular	
-2	$(5, 8, 6, 4, 2, 0, \tfrac{37}{2}\sqrt{2})$	κ_2 in Table 3	$27 - 2 = 25$

(Since $d^- \neq d^+$, by Theorem 4.10, or rather by its proof, $\mu(\nu)$ cannot be regular.)

Example 5.48 ($\mathbb{C}$-simple type E_7 with $r = 1$, noncohomological weight). Suppose $\nu = (-7, 5, 5, 2, 1, 0, 3\sqrt{2})$, so that

$$\begin{aligned}\lambda^{(0)} = \lambda + \rho_{\mathrm{G}} &= (-2, 4, 4, 3, 2, 1, 6\sqrt{2}) + (5, 4, 3, 2, 1, 0, \tfrac{17}{2}\sqrt{2}) \\ &= (3, 8, 7, 5, 3, 1, \tfrac{29}{2}\sqrt{2}).\end{aligned}$$

Note that $\lambda^{(0)}$ is not regular because there is no unique maximal pairing between it and the weights in Tables 3 and 4, and hence ν is not cohomological in the sense of Definition 2.6. Then $[d^-, d^+] = [23, 24]$ by calculations summarized as follows:

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d - l^{(s)} = d^{(s)}$
1	$(4, 8, 7, 5, 3, 1, 15\sqrt{2})$	κ_3 in Table 3	$27 - 3 = 24$
-1	$(2, 8, 7, 5, 3, 1, 14\sqrt{2})$	κ_4 in Table 3	$27 - 4 = 23$

Example 5.49 (restriction of scalar of type E_7). Suppose the group is $\mathbb{Q}$-simple but has two $\mathbb{C}$-simple factors of E_7, which we denote by $\upsilon = 1$ and 2, respectively, such that $r_1 = r_2 = 1$, and so that $d = d_1 + d_2 = 27 + 27 = 54$. Suppose $\nu = (\nu_1, \nu_2)$, where ν_1 is the ν in Example 5.47, and where ν_2 is the ν in Example 5.48, so that

$$\lambda^{(0)} = ((7, 8, 6, 4, 2, 0, \tfrac{39}{2}\sqrt{2}), (3, 8, 7, 5, 3, 1, \tfrac{29}{2}\sqrt{2})).$$

Then $[d^-, d^+] = [28, 52]$ by calculations summarized as follows (with the two factors of each weight in the column of $\kappa^{(s)}$ can be found in Tables 3 and 4):

s	$\lambda^{(s)}$	$\kappa^{(s)}$	$d^{(s)}$
1	$((8, 8, 6, 4, 2, 0, 20\sqrt{2}), (4, 8, 7, 5, 3, 1, 15\sqrt{2}))$	not regular	
2	$((9, 8, 6, 4, 2, 0, \tfrac{41}{2}\sqrt{2}), (5, 8, 7, 5, 3, 1, \tfrac{31}{2}\sqrt{2}))$	not regular	
3	$((10, 8, 6, 4, 2, 0, 21\sqrt{2}), (6, 8, 7, 5, 3, 1, 15\sqrt{2}))$	(κ_0, κ_2)	52
-1	$((6, 8, 6, 4, 2, 0, 19\sqrt{2}), (2, 8, 7, 5, 3, 1, 14\sqrt{2}))$	not regular	
-2	$((5, 8, 6, 4, 2, 0, \tfrac{37}{2}\sqrt{2}), (1, 8, 7, 5, 3, 1, \tfrac{27}{2}\sqrt{2}))$	not regular	
-3	$((4, 8, 6, 4, 2, 0, 18\sqrt{2}), (0, 8, 7, 5, 3, 1, 13\sqrt{2}))$	not regular	
-4	$((3, 8, 6, 4, 2, 0, \tfrac{35}{2}\sqrt{2}), (-1, 8, 7, 5, 3, 1, \tfrac{25}{2}\sqrt{2}))$	not regular	
-5	$((2, 8, 6, 4, 2, 0, 17\sqrt{2}), (-2, 8, 7, 5, 3, 1, 12\sqrt{2}))$	not regular	
-6	$((1, 8, 6, 4, 2, 0, \tfrac{33}{2}\sqrt{2}), (-3, 8, 7, 5, 3, 1, \tfrac{23}{2}\sqrt{2}))$	not regular	
-7	$((0, 8, 6, 4, 2, 0, 16\sqrt{2}), (-4, 8, 7, 5, 3, 1, 11\sqrt{2}))$	not regular	
-8	$((-1, 8, 6, 4, 2, 0, \tfrac{31}{2}\sqrt{2}), (-5, 8, 7, 5, 3, 1, \tfrac{21}{2}\sqrt{2}))$	not regular	
-9	$((-2, 8, 6, 4, 2, 0, 15\sqrt{2}), (-6, 8, 7, 5, 3, 1, 10\sqrt{2}))$	not regular	
-10	$((-3, 8, 6, 4, 2, 0, \tfrac{29}{2}\sqrt{2}), (-7, 8, 7, 5, 3, 1, \tfrac{19}{2}\sqrt{2}))$	not regular	
-11	$((-4, 8, 6, 4, 2, 0, 14\sqrt{2}), (-8, 8, 7, 5, 3, 1, 9\sqrt{2}))$	not regular	
-12	$((-5, 8, 6, 4, 2, 0, \tfrac{27}{2}\sqrt{2}), (-9, 8, 7, 5, 3, 1, \tfrac{17}{2}\sqrt{2}))$	not regular	
-13	$((-6, 8, 6, 4, 2, 0, 13\sqrt{2}), (-10, 8, 7, 5, 3, 1, 8\sqrt{2}))$	not regular	
-14	$((-7, 8, 6, 4, 2, 0, \tfrac{25}{2}\sqrt{2}), (-11, 8, 7, 5, 3, 1, \tfrac{15}{2}\sqrt{2}))$	$(\kappa_{10_{\mathrm{II}}}, \kappa_{16_{\mathrm{I}}})$	28

If the two $\mathbb{C}$-simple factors were not in the same $\mathbb{Q}$-simple factor, then the two factors $\nu_1 = (7, 8, 6, 4, 2, 0, \frac{39}{2}\sqrt{2})$ and $\nu_2 = (3, 8, 7, 5, 3, 1, \frac{29}{2}\sqrt{2})$ would have defined the intervals $[25, 27]$ and $[23, 24]$ on their respective factors, whose end points sum up to those of $[48, 51]$, which is much narrower than $[28, 52]$.

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References

1. A. Ash, D. Mumford, M. Rapoport, and Y. Tai, *Smooth compactification of locally symmetric varieties*, Lie Groups: History Frontiers and Applications, vol. 4, Math Sci Press, Brookline, Massachusetts, 1975, TEXified and republished as [2].
2. ———, *Smooth compactification of locally symmetric varieties*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, New York, 2010.
3. W. L. Baily, Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. Math. (2) **84** (1966), no. 3, 442–528.
4. A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, preprint, available online.
5. I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, *Differential operators on the base affine space and a study of $\mathfrak{g}$-modules*, in Gelfand [16], pp. 21–64.
6. A. Borel, *Some finiteness properties of adele groups over number fields*, Publ. Math. Inst. Hautes Étud. Sci. **16** (1963), 5–30.
7. L. Clozel and J. S. Milne (eds.), *Automorphic forms, Shimura varieties, and L-functions. Volume I*, Perspectives in Mathematics, vol. 10, Proceedings of a Conference held at the University of Michigan, Ann Arbor, July 6–16, 1988, Academic Press Inc., Boston, 1990.
8. L. Clozel and J. S. Milne (eds.), *Automorphic forms, Shimura varieties, and L-functions. Volume II*, Perspectives in Mathematics, vol. 11, Proceedings of a Conference held at the University of Michigan, Ann Arbor, July 6–16, 1988, Academic Press Inc., Boston, 1990.
9. Congrès International des Mathématiciens, 1/10 Septembre 1970, Nice, France, *Actes du Congrès International des Mathématiciens, 1970, publiés sous la direction du Comité d'Organisation du Congrès*, vol. 1, Gauthier-Villars, Paris, 1971.
10. P. Deligne, *Equations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, vol. 163, Springer-Verlag, Berlin, Heidelberg, New York, 1970.
11. P. Deligne and L. Illusie, *Relèvements modulo p^2 et décompositions du complex de de Rham*, Invent. Math. **89** (1987), 247–270.
12. H. Esnault and E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar, vol. 20, Birkhäuser, Boston, 1992.
13. G. Faltings, *On the cohomology of locally symmetric hermitian spaces*, in Malliavin [39], pp. 55–98.
14. G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 22, Springer-Verlag, Berlin, Heidelberg, New York, 1990.
15. J. Franke, *Harmonic analysis in weighted L_2-spaces*, Ann. Sci. Ecole Norm. Sup. (4) **31** (1998), 181–279.
16. I. M. Gelfand (ed.), *Lie groups and their representations*, Summer School of the Bolyai János Mathematical Society, (Budapest, 1971), Adam Hilger Ltd., London, 1975.
17. R. Godement, *Topologie algébrique et théorie des faisceaux*, Actualités scientifiques et industrielles, vol. 1252, Hermann, Paris, 1964.
18. A. Grothendieck (ed.), *Cohomologie locale des faiseaux cohérents et théorèmes de Lefschetz locaux et globaux* (*SGA 2*), North-Holland Publishing Co., Amsterdam, 1968.
19. A. Grothendieck (ed.), *Revêtements étales et groupe fondamental* (*SGA 1*), Lecture Notes in Mathematics, vol. 224, Springer-Verlag, Berlin, Heidelberg, New York, 1971.
20. M. Harris, *Functorial properties of toroidal compactifications of locally symmetric varieties*, Proc. London Math. Soc. (3) **59** (1989), 1–22.
21. ———, *Automorphic forms and the cohomology of vector bundles on Shimura varieties*, in Clozel and Milne [8], pp. 41–91.
22. M. Harris and S. Zucker, *Boundary cohomology of Shimura varieties III. Coherent cohomology on higher-rank boundary strata and applications to Hodge theory*, Mémoires de la Société Mathématique de France. Nouvelle Série, vol. 85, Société Mathématique de France, Paris, 2001.
23. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, Rhode Island, 2001.
24. J.-I. Igusa (ed.), *Algebraic analysis, geometry, and number theory*, Proceedings of the JAMI Inaugural Conference, The Johns Hopkins University Press, Baltimore, 1989.
25. Luc Illusie, *Réduction semi-stable et décomposition de complexes de de Rham*, Duke Math. J. **60** (1990), no. 1, 139–185.

26. V. G. Kac, *Infinite dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, New York, 1990.
27. K. Kato, *Logarithmic structures of Fontaine–Illusie*, in Igusa [24], pp. 191–224.
28. N. M. Katz, *The regularity theorem in algebraic geometry*, in *Actes du Congrès International des Mathématiciens, 1970, publiés sous la direction du Comité d'Organisation du Congrès* [9], pp. 437–443.
29. G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings I*, Lecture Notes in Mathematics, vol. 339, Springer-Verlag, Berlin, Heidelberg, New York, 1973.
30. K.-W. Lan, *Comparison between analytic and algebraic constructions of toroidal compactifications of PEL-type Shimura varieties*, J. Reine Angew. Math. **664** (2012), 163–228.
31. ———, *Toroidal compactifications of PEL-type Kuga families*, Algebra Number Theory **6** (2012), no. 5, 885–966.
32. ———, *Arithmetic compactification of PEL-type Shimura varieties*, London Mathematical Society Monographs, vol. 36, Princeton University Press, Princeton, 2013, errata and revision available online at the author's website.
33. ———, *Higher Koecher's principle*, Math. Res. Lett. **23** (2016), no. 1, 163–199.
34. K.-W. Lan and B. Stroh, *Relative cohomology of cuspidal forms on PEL-type Shimura varieties*, Algebra Number Theory **8** (2014), no. 8, 1787–1799.
35. K.-W. Lan and J. Suh, *Liftability of mod p cusp forms of parallel weights*, Int. Math. Res. Not. IMRN **2011** (2011), 1870–1879.
36. ———, *Vanishing theorems for torsion automorphic sheaves on compact PEL-type Shimura varieties*, Duke Math. J. **161** (2012), no. 6, 1113–1170.
37. ———, *Vanishing theorems for torsion automorphic sheaves on general PEL-type Shimura varieties*, Adv. Math. **242** (2013), 228–286.
38. J.-S. Li and J. Schwermer, *On the Eisenstein cohomology of arithmetic groups*, Duke Math. J. **123** (2004), no. 1, 141–169.
39. M.-P. Malliavin (ed.), *Séminaire d'algèbre Paul Dubreil et Marie–Paule Malliavin*, Lecture Notes in Mathematics, vol. 1029, Proceedings, Paris 1982 (35ème Année), Springer-Verlag, Berlin, Heidelberg, New York, 1983.
40. J. S. Milne, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, in Clozel and Milne [7], pp. 283–414.
41. J. S. Milne and J. Suh, *Nonhomeomorphic conjugates of connected Shimura varieties*, Amer. J. Math. **132** (2010), no. 3, 731–750.
42. D. Mumford, *Hirzebruch's proportionality theorem in the non-compact case*, Invent. Math. **42** (1977), 239–272.
43. A. Ogus, *F-crystals, Griffiths transversality, and the Hodge decomposition*, Astérisque, no. 221, Société Mathématique de France, Paris, 1994.
44. M. Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 2, 221–333.
45. J.-P. Serre, *Geométrie algébrique et géométrie analytique*, Ann. Inst. Fourier. Grenoble **6** (1955–1956), 1–42.
46. Y.-T. Siu, *Analytic sheaves of local cohomology*, Trans. Amer. Math. Soc. **148** (1970), 347–366.
47. T. A. Springer, *Linear algebraic groups*, 2nd ed., Progress in Mathematics, vol. 9, Birkhäuser, Boston, 1998.
48. B. Stroh, *Relèvement de formes modulaires de Siegel*, Proc. Amer. Math. Soc. **138** (2010), 3089–3094.
49. ———, *Classicité en théorie de Hida*, Amer. J. Math. **135** (2013), no. 4, 861–889.
50. J. Suh, *Vanishing theorems for mixed hodge modules and applications*, to appear in J. Eur. Math. Soc., available upon request.
51. D. A. Vogan, Jr. and G. J. Zuckerman, *Unitary representations with non-zero cohomology*, Compositio Math. **53** (1984), no. 1, 51–90.
52. X. Zhu, *On the coherence conjecture of Pappas and Rapoport*, Ann. Math. (2) **180** (2014), no. 1, 1–85.

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