COHOMOLOGY OF AUTOMORPHIC BUNDLES

KAI-WEN LAN

ABSTRACT. In this survey article, we review some recent works (by the author and his collaborators Junecue Suh, Michael Harris, Richard Taylor, Jack Thorne, and Benoît Stroh) on the cohomology of automorphic bundles over locally symmetric varieties and some related geometric objects.

CONTENTS

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1. Classical story: modular curves and modular forms

Let us begin by briefly reviewing the classical story of modular curves and modular forms. For more details and references, we shall refer the readers to the first two sections of the survey article [Lan12b], which had a similar starting point. Nevertheless, our goals in this article are more general, with less emphasis on the good reduction integral models of PEL-type Shimura varieties.

1.1. **Classical modular forms.** The group $\mathrm{SL}_2(\mathbb{R})$ acts on the *Poincaré upper-half plane*

$$\mathcal{H} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$$

by the usual *Möbius transformation*

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : z \mapsto \gamma(z) := \frac{az+b}{cz+d}.$$

This is induced by the (transitive) action of $\mathrm{SL}_2(\mathbb{C})$ on the *projective coordinates* of $\mathbb{P}^1(\mathbb{C})$:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) : \begin{pmatrix} z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \sim \begin{pmatrix} \gamma(z) \\ 1 \end{pmatrix}.$$

Note that $\sim$ is given by division by the factor $(cz+d)$.

Suppose $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a finite index subgroup (which is then an *arithmetic subgroup* of $\mathrm{SL}_2(\mathbb{Q})$). Suppose $k \in \mathbb{Z}$ is an integer.

Definition 1.1. *A* **classical modular form** *of level* Γ *and weight* k *is a holomorphic function* $f : \mathcal{H} \to \mathbb{C}$ *satisfying the following two conditions:*

(1) (**automorphy condition**) $f(\gamma(z)) = (cz+d)^k f(z)$ *for all* $\gamma \in \Gamma$.
(2) (**growth condition**) $(cz+d)^{-k} f(\gamma(z))$ *stays bounded as* $\mathrm{im}(z) \to +\infty$*, for all* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

We say that f *is a* **cusp form** *if, instead of* (2)*, it satisfies the following:*

(3) (**cuspidal condition**) $(cz+d)^{-k} f(\gamma(z)) \to 0$ *as* $\mathrm{im}(z) \to +\infty$*, for all* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

We shall denote the space of modular forms (resp. cusp forms) of level Γ and weight k by $M_k(\Gamma; \mathbb{C})$ (resp. $S_k(\Gamma; \mathbb{C})$). (These might be zero spaces.)

1.2. **Modular curves and a geometric definition of modular forms.** We can redefine $M_k(\Gamma; \mathbb{C})$ and $S_k(\Gamma; \mathbb{C})$ geometrically. The group Γ acts naturally on $\mathcal{H}$ and on

$$\mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}),$$

and we have the open and compactified modular curves

$$Y_\Gamma := \Gamma\backslash\mathcal{H} \hookrightarrow X_\Gamma := \Gamma\backslash\mathcal{H}^* \tag{1.2}$$

(with a suitable topology on $\Gamma\backslash\mathcal{H}^*$). The pullback of $\mathcal{O}_{\mathbb{P}^1(\mathbb{C})}(1)$ to $\mathcal{H}$ descends to a line bundle ω on Y_Γ, which extends to a line bundle ω on X_Γ such that

$$M_k(\Gamma; \mathbb{C}) \cong H^0(X_\Gamma, \omega^{\otimes k}) \tag{1.3}$$

and

$$S_k(\Gamma; \mathbb{C}) \cong H^0(X_\Gamma, \omega^{\otimes k}(-\infty)), \tag{1.4}$$

where $(-\infty)$ means vanishing at the *cusps* $X_\Gamma - Y_\Gamma = \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$.

By the *Kodaira–Spencer isomorphism* and *Serre duality*, we have

$$\omega^2(-\infty) \cong \Omega^1_{X_\Gamma/\mathbb{C}}$$

and

$$\overline{S_k(\Gamma;\mathbb{C})} \cong \overline{H^0(X_\Gamma, \omega^{\otimes k}(-\infty))} \cong H^1(X_\Gamma, \omega^{\otimes(2-k)}),$$

where the complex conjugations (denoted by overlines) are induced by integrations on modular curves. When $k \geq 2$, we have the *Eichler–Shimura isomorphism*

$$H^1(Y_\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^{\oplus 2})) \cong M_k(\Gamma;\mathbb{C}) \oplus \overline{S_k(\Gamma;\mathbb{C})}, \tag{1.5}$$

where $\mathrm{Sym}^{k-2}(\mathbb{C}^{\oplus 2})$ abusively denotes the local system (over Y_Γ) associated with the algebraic representation (denoted by the same symbols) of SL_2 over $\mathbb{C}$ (induced by the standard action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$), which can be rewritten as

$$H^1(Y_\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^{\oplus 2})) \cong H^0(X_\Gamma, \omega^{\otimes k}) \oplus H^1(X_\Gamma, \omega^{\otimes(2-k)}). \tag{1.6}$$

When $k = 2$, this is a consequence of the comparison isomorphism

$$H^1(Y_\Gamma, \mathbb{C}) \cong H^1(X_\Gamma, \Omega^\bullet_{X_\Gamma/\mathbb{C}}(\log\infty))$$

(where the right-hand side denotes the first hypercohomology of the log de Rham complex $\Omega^\bullet_{X_\Gamma/\mathbb{C}}(\log\infty)$), and the degeneration of Hodge spectral sequence:

$$H^1(X_\Gamma, \Omega^\bullet_{X_\Gamma/\mathbb{C}}(\log\infty)) \cong H^0(X_\Gamma, \Omega^1_{X_\Gamma/\mathbb{C}}(\log\infty)) \oplus H^1(X_\Gamma, \mathscr{O}_{X_\Gamma}).$$

The previous isomorphism (1.6) for general $k > 2$ is a similar consequence for nontrivial coefficient systems (using mixed Hodge theory).

1.3. **Integral models and some applications.** In this subsection, let us assume that Γ is a *congruence subgroup* of $\mathrm{SL}_2(\mathbb{Z})$, which means Γ contains the *principal congruence subgroup*

$$\Gamma(N) := \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$$

for some integer $N \geq 1$. For any rational prime p and any field isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$, we can compare $H^1(Y_\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^{\oplus 2}))$ with $H^1_{\text{ét}}(Y_\Gamma, \mathrm{Sym}^{k-2}(\overline{\mathbb{Q}}_p^{\oplus 2}))$, where $\mathrm{Sym}^{k-2}(\overline{\mathbb{Q}}_p^{\oplus 2})$ abusively denotes the (lisse) $\overline{\mathbb{Q}}_p$-étale sheaf associated with the algebraic representation (denoted by the same symbols) of SL_2 over $\overline{\mathbb{Q}}_p$, and such a comparison is compatible with varying Γ (and with the *Hecke actions* on these spaces). An important fact is that Y_Γ admits a model over some number field K, and hence $\mathrm{Gal}(\overline{K}/K)$ acts on $H^1_{\text{ét}}(Y_\Gamma, \mathrm{Sym}^{k-2}(\overline{\mathbb{Q}}_p^{\oplus 2}))$. Very roughly speaking (we are being intentionally vague here), this provides the main source of *Galois representations* attached to cusp forms of weight $k \geq 2$. By using also some *integral model* of X_Γ (over the integers $\mathcal{O}_K$), we obtain information about the restriction of such Galois representations to $\mathrm{Gal}(\overline{K}_v/K_v)$ at nonarchimedean places v. By using integral models differently, we can define the integral versions $M_k(\Gamma;\mathcal{O}_K)$ and $S_k(\Gamma;\mathcal{O}_K)$, and study *congruences* between modular forms. Then we can extend the attachment of Galois representations to the case of the *low weight* $k = 1$ (see [DS74]). (For simplicity, we mentioned only the semisimple algebraic group SL_2 in the above, but for the consideration of Hecke and Galois actions, it is important to also introduce the reductive algebraic group GL_2.)

In what follows, our goals will be to explain the generalizations of modular curves and modular forms to higher dimensions, and to provide a survey of some recent results on such generalizations due to this author and his collaborators Junecue Suh, Michael Harris, Richard Taylor, Jack Thorne, and Benoît Stroh.

2. Locally symmetric varieties and automorphic bundles

Here is an overview of the generalizations we shall explain:

classical story	generalizations
SL_2 or GL_2	more general algebraic groups
Γ	arithmetic subgroups
Poincaré upper-half plane $\mathcal{H}$	Hermitian symmetric domains
open modular curve $\Gamma\backslash\mathcal{H}$	locally symmetric varieties or Shimura varieties
compactified modular curve $\Gamma\backslash\mathcal{H}^*$	various compactifications
ω^k and $\mathrm{Sym}^k(\mathbb{C}^{\oplus 2})$	automorphic (vector) bundles (two kinds)
M_k = sections of ω^k	sections and also cohomology of automorphic bundles
Eichler–Shimura isomorphism	Faltings's dual BGG spectral sequence (degeneration by mixed Hodge theory)
integral models	integral models
⋮	⋮

2.1. **Locally symmetric varieties and their compactifications.** Let us start with some generalization of the Poincaré upper-half plane. For simplicity of exposition, consider the following setup:

- $\mathcal{H}$ is a *Hermitian symmetric domain* of dimension d, and
- G is a simply-connected connected semisimple algebraic group over $\mathbb{Q}$

such that $\mathcal{H} \cong \mathrm{G}(\mathbb{R})/K$ for some maximal compact subgroup K of G. (The setup here can be generalized to the case where $\mathcal{H}$ is a finite disjoint union of Hermitian symmetric domains with a transitive action of $\mathrm{G}(\mathbb{R})$, where G is a possibly disconnected reductive algebraic group satisfying some conditions. This more general setup will be tacitly allowed when we discuss about Shimura varieties and their integral models.) Then there is the *Borel embedding*

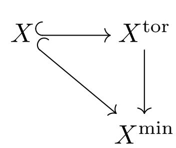
$$\mathcal{H} = \mathrm{G}(\mathbb{R})/K \hookrightarrow \mathcal{H}^\vee = \mathrm{G}(\mathbb{C})/\mathrm{P}(\mathbb{C})$$

for some maximal parabolic subgroup P of $\mathrm{G}_\mathbb{C}$, with a Levi subgroup M such that $\mathrm{M}(\mathbb{C}) = K_\mathbb{C}$ in $\mathrm{G}(\mathbb{C})$. (See, e.g., [Hel01, Ch. III, Sec. 7], [AMRT75, Ch. III, Sec. 2.1], and [Mil90, Sec. III.1].) This generalizes the canonical embedding of the Poincaré upper-half plane into the projective line $\mathbb{P}^1(\mathbb{C})$.

Let Γ be any arithmetic subgroup of $\mathrm{G}(\mathbb{Q})$ which we assume to be *neat* (so that it acts freely on $\mathcal{H}$), and let

$$X := \Gamma\backslash\mathcal{H}.$$

Then we have some useful compactifications of X, generalizing (1.2) in two ways:



In this diagram, the *minimal compactification* $X^{\min}$ due to Satake and Baily–Borel (see [BB66]) is canonical, normal, and projective. This shows that X is quasi-projective and canonically algebraic. But the generally rather singular $X^{\min}$ is not a good starting point for generalizing modular forms. On the other hand, the *toroidal compactifications* X^{tor} due to Mumford and others (see [AMRT75] and [AMRT10]) are noncanonical, but still canonically depend on certain *cone decompositions* (or *fans*), and can be chosen to be projective and smooth with boundaries given by simple normal crossings divisors. For simplicity, we shall always assume that the toroidal compactifications we consider have these nice properties. (This is harmless for our purpose.) They are useful for applications of mixed Hodge theory, and for defining and studying generalizations of modular forms.

2.2. **Automorphic bundles and canonical extensions.** To define the desired generalizations of modular forms, we need the automorphic (vector) bundles over X, together with their canonical and subcanonical extensions over X^{tor}.

Let $\mathrm{Rep}_{\mathbb{C}}(\mathrm{G}_{\mathbb{C}})$ (resp. $\mathrm{Rep}_{\mathbb{C}}(\mathrm{P})$, resp. $\mathrm{Rep}_{\mathbb{C}}(\mathrm{M})$) denote the category of finite dimensional representations of $\mathrm{G}_{\mathbb{C}}$ (resp. P, resp. M) over $\mathbb{C}$. Given any object W in $\mathrm{Rep}_{\mathbb{C}}(\mathrm{P})$, we can naturally define a vector bundle

$$\underline{W} := (\mathrm{G}(\mathbb{C}) \times W)/\mathrm{P}(\mathbb{C})$$

over $\mathcal{H}^\vee = \mathrm{G}(\mathbb{C})/\mathrm{P}(\mathbb{C})$, which pulls back to $\mathcal{H}$ and descends to $X = \Gamma\backslash\mathcal{H}$, still denoted by $\underline{W}$. By abuse of notation, we shall denote the corresponding sheaf of sections over X by the same symbol $\underline{W}$. Given any object W in $\mathrm{Rep}_{\mathbb{C}}(\mathrm{M})$, we can pull it back to an object W in $\mathrm{Rep}_{\mathbb{C}}(\mathrm{P})$ via the canonical homomorphism $\mathrm{P} \twoheadrightarrow \mathrm{M}$ and define a vector bundle $\underline{W}$ (and its sheaf of sections) over X. On the other hand, given any object V in $\mathrm{Rep}_{\mathbb{C}}(\mathrm{G}_{\mathbb{C}})$, its restriction $V|_{\mathrm{P}}$ defines a vector bundle $\underline{V} := \underline{(V|_{\mathrm{P}})}$ over X, and the $\operatorname{Lie} \mathrm{G}_{\mathbb{C}}$ action on V defines an integrable connection

$$\nabla : \underline{V} \to \underline{V} \otimes_{\mathscr{O}_X} \Omega^1_{X/\mathbb{C}}. \tag{2.1}$$

Such $\underline{W}$ and $(\underline{V}, \nabla)$ will be called *automorphic bundles* over X.

According to Mumford and Harris (see [Mum77] and [Har89]), we have a *canonical extension*

$$\underline{W}^{\mathrm{can}}$$

of $\underline{W}$ to a vector bundle over X^{tor}, for each $\underline{W}$ as above. As explained in [Har89, Sec. 4], Deligne's canonical extension $(\underline{V}^{\mathrm{can}}, \nabla^{\mathrm{can}})$ of $(\underline{V}, \nabla)$ over X^{tor} (see [Del70]) is compatible with the above in the sense that $\underline{V}^{\mathrm{can}} = \underline{(V|_{\mathrm{P}})}^{\mathrm{can}}$. Following [Har90, Sec. 2], we introduce the *subcanonical extensions*

$$\underline{W}^{\mathrm{sub}} := \underline{W}^{\mathrm{can}}(-D)$$

and

$$\underline{V}^{\mathrm{sub}} := \underline{V}^{\mathrm{can}}(-D)$$

of $\underline{W}$ and $\underline{V}$ over X^{tor}, respectively, where

$$D := (X^{\mathrm{tor}} - X)_{\mathrm{red}}$$

is the (reduced) boundary divisor. Then we obtain integrable connections

$$\nabla : \underline{V}^{\mathrm{can}} \to \underline{V}^{\mathrm{can}} \otimes_{\mathscr{O}_{X^{\mathrm{tor}}}} \Omega^1_{X^{\mathrm{tor}}/\mathbb{C}}(\log D)$$

and

$$\nabla : \underline{V}^{\mathrm{sub}} \to \underline{V}^{\mathrm{sub}} \otimes_{\mathscr{O}_{X^{\mathrm{tor}}}} \Omega^1_{X^{\mathrm{tor}}/\mathbb{C}}(\log D)$$

with *log poles* along D, extending (2.1), for each $\underline{V}$ as above.

The coherent sheaf cohomology groups $H^i(X^{\mathrm{tor}}, \underline{W}^{\mathrm{can}})$ and $H^i(X^{\mathrm{tor}}, \underline{W}^{\mathrm{sub}})$ are called the *coherent cohomology* of X^{tor} of weight W, which are natural generalizations of modular forms (cf. (1.3) and (1.4)). (Such cohomology depends only on X and W, but not on the choice of X^{tor}.) The hypercohomology of the de Rham complex of $\underline{V}^{\mathrm{can}}$ (resp. $\underline{V}^{\mathrm{sub}}$), defined by the connection above with log poles, computes the de Rham cohomology $H^{\bullet}_{\mathrm{dR}}(X, \underline{V})$ (resp. the compactly supported $H^{\bullet}_{\mathrm{dR,c}}(X, \underline{V})$).

2.3. **Cohomological weights and dual BGG decompositions.** Let us compatibly choose weights $\mathbf{X}_{\mathrm{G}_\mathbb{C}} = \mathbf{X}_{\mathrm{M}}$ and positive roots $\Phi^+_{\mathrm{G}_\mathbb{C}}$ and Φ^+_{M}. Then $\Phi^+_{\mathrm{G}_\mathbb{C}} \supset \Phi^+_{\mathrm{M}}$ and $\mathbf{X}^+_{\mathrm{G}_\mathbb{C}} \subset \mathbf{X}^+_{\mathrm{M}}$. For each $\mu \in \mathbf{X}^+_{\mathrm{G}_\mathbb{C}}$ and $\nu \in \mathbf{X}^+_{\mathrm{M}}$, we shall denote by $V_\mu \in \mathrm{Rep}_\mathbb{C}(\mathrm{G}_\mathbb{C})$ and $W_\nu \in \mathrm{Rep}_\mathbb{C}(\mathrm{M})$ the irreducible representations of highest weights μ and ν, respectively. For simplicity, we shall make various minor adjustments in the notation and terminologies without explicitly introducing them. For example, we shall say that $H^i(X^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_\nu)$ and $H^i(X^{\mathrm{tor}}, \underline{W}^{\mathrm{sub}}_\nu)$ are coherent cohomology *of weight* ν, rather than of weight W_ν.

Consider $\rho = \rho_{\mathrm{G}_\mathbb{C}} := \frac{1}{2}\sum_{\alpha \in \Phi^+_{\mathrm{G}_\mathbb{C}}} \alpha$, $\rho_{\mathrm{M}} := \frac{1}{2}\sum_{\alpha \in \Phi^+_{\mathrm{M}}} \alpha$, and $\rho^{\mathrm{M}} := \rho - \rho_{\mathrm{M}}$. Consider the Weyl groups $W = W_{\mathrm{G}_\mathbb{C}} \supset W_{\mathrm{M}}$, and consider the minimal length representatives $W^{\mathrm{M}} := \{w \in W : w(X^+_{\mathrm{G}_\mathbb{C}}) \subset X^+_{\mathrm{M}}\}$ of $W_{\mathrm{M}} \backslash W$. Consider the usual *dot action* of W on $\mathbf{X}$ given by $w \cdot \mu = w(\mu + \rho) - \rho$.

Definition 2.2. *We say* $\nu \in X^+_{\mathrm{M}}$ *is* (*de Rham*) **cohomological** *if there exist* $\mu = \mu(\nu) \in \mathbf{X}^+_{\mathrm{G}_\mathbb{C}}$ *and* $w = w(\nu) \in W^{\mathrm{M}}$ *such that* $W_\nu \cong W^\vee_{w\cdot\mu}$.

This notion is justified by Faltings's *dual BGG spectral sequence* (see [Fal83, Sec. 3 and 7] and [FC90, Ch. VI, Sec. 5]) and its degeneracy due to the theory of *mixed Hodge modules* (see [Sai90] and [HZ01, Cor. 4.2.3]):

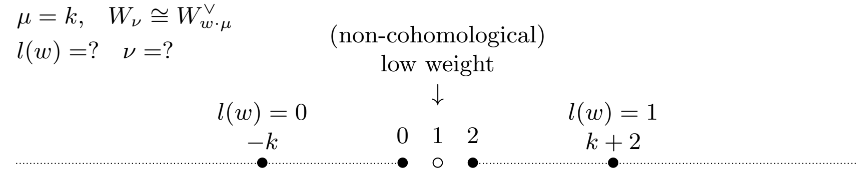
$$H^i_{\mathrm{dR}}(X, \underline{V}^\vee_\mu) \cong \bigoplus_{w \in W^{\mathrm{M}}} H^{i-l(w)}(X^{\mathrm{tor}}, (\underline{W}^{\mathrm{can}}_{w\cdot\mu})^\vee) \tag{2.3}$$

and

$$H^i_{\mathrm{dR,c}}(X, \underline{V}^\vee_\mu) \cong \bigoplus_{w \in W^{\mathrm{M}}} H^{i-l(w)}(X^{\mathrm{tor}}, (\underline{W}^{\mathrm{sub}}_{w\cdot\mu})^\vee). \tag{2.4}$$

We shall call such isomorphisms *dual BGG decompositions*. The de Rham cohomology at the left-hand sides of (2.3) and (2.4) can be compared with the Betti (singular) and étale cohomology with coefficients in the local systems given by the corresponding sheaves of horizontal sections, while the coherent cohomology on the right-hand sides of (2.3) and (2.4) can be viewed as generalizations of modular forms. But only coherent cohomology of cohomological weights can contribute to the de Rham cohomology via the dual BGG decompositions.

Example 2.5 ($\mathrm{G} = \mathrm{SL}_2$; modular curve case).

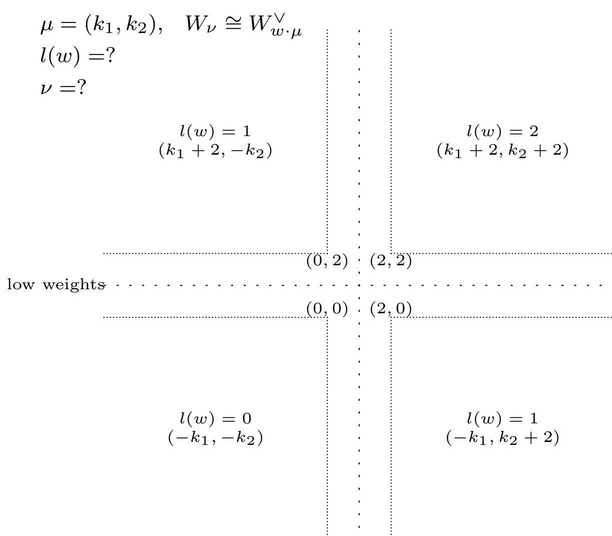


In this case, since $\underline{V}_k = \underline{V}_k^\vee = \mathrm{Sym}^k(\mathbb{C}^{\oplus 2})$ for $k \geq 0$ and $\underline{W}_k^{\mathrm{can}} = (\underline{W}_{-k}^{\mathrm{can}})^\vee = \omega^k$ for all k, the dual BGG decomposition (2.3) can be identified with the Eichler–Shimura isomorphism (1.6) (for weights ≥ 2):

$$H^1_{\mathrm{dR}}(X, \underline{V}_k) \cong H^0(X^{\mathrm{tor}}, \underline{W}_{k+2}^{\mathrm{can}}) \oplus H^1(X^{\mathrm{tor}}, \underline{W}_{-k}^{\mathrm{can}}).$$

Note that *weight* 1 *modular forms* do not contribute to the de Rham cohomology (but can be studied by congruences with forms of cohomological weights).

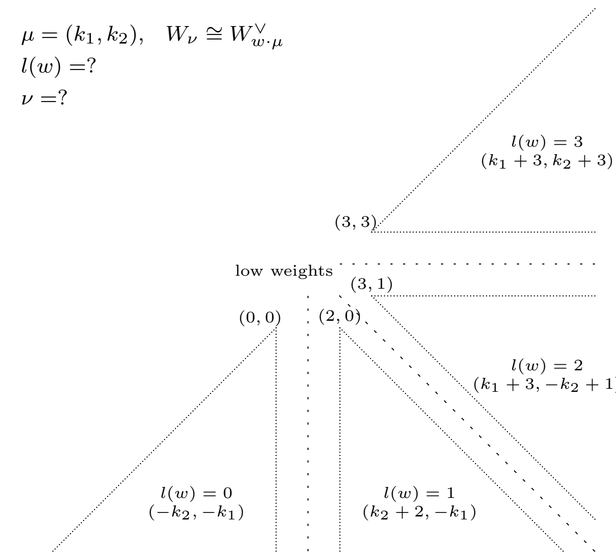
Example 2.6 ($\mathrm{G} = \mathrm{SL}_{2,F}$ with $F/\mathbb{Q}$ real quadratic; Hilbert modular surfaces).



In this case, the dual BGG decomposition gives

$$\begin{aligned} H^2_{\mathrm{dR}}(X, \underline{V}_{(k_1,k_2)}) \cong & H^0(X^{\mathrm{tor}}, \underline{W}_{(k_1+2,k_2+2)}^{\mathrm{can}}) \oplus H^1(X^{\mathrm{tor}}, \underline{W}_{(-k_1,k_2+2)}^{\mathrm{can}}) \\ & \oplus H^1(X^{\mathrm{tor}}, \underline{W}_{(k_1+2,-k_2)}^{\mathrm{can}}) \oplus H^2(X^{\mathrm{tor}}, \underline{W}_{(-k_1,-k_2)}^{\mathrm{can}}). \end{aligned}$$

Example 2.7 ($\mathrm{G} = \mathrm{Sp}_4$; Siegel threefolds).



In this case, the dual BGG decomposition gives

$$\begin{aligned} H^3_{\mathrm{dR}}(X, \underline{V}_{(k_1,k_2)}) \cong & H^0(X^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_{(k_1+3,k_2+3)}) \oplus H^1(X^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_{(k_1+3,-k_2+1)}) \\ & \oplus H^2(X^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_{(k_2+2,-k_1)}) \oplus H^3(X^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_{(-k_2,-k_1)}) \end{aligned}$$

3. Vanishing theorems

In this section, we present some vanishing theorems for the cohomology of automorphic bundles over general locally symmetric varieties, in the author's collaborations with Junecue Suh and Benoît Stroh, together with some examples.

3.1. **Positivity of automorphic line bundles.** By [BB66], the canonical bundle $\underline{W}_{2\rho^{\mathrm{M}}} \cong \Omega^d_{X/\mathbb{C}}$ of X, where $d = \dim_{\mathbb{C}}(X)$, is *ample*. By [Mum77, Prop. 3.4 b)], $\underline{W}^{\mathrm{can}}_{2\rho^{\mathrm{M}}} \cong \Omega^d_{X^{\mathrm{tor}}/\mathbb{C}}(\log D)$, where $D := (X^{\mathrm{tor}} - X)_{\mathrm{red}}$, descends to an ample line bundle over $X^{\min}$. Hence, although the automorphic line bundle $\underline{W}_{2\rho^{\mathrm{M}}}$ over X is ample, its canonical extension $\underline{W}^{\mathrm{can}}_{2\rho^{\mathrm{M}}}$ over X^{tor} is not ample unless the canonical morphism $X^{\mathrm{tor}} \to X^{\min}$ is an isomorphism.

One might guess that the subcanonical extension $\underline{W}^{\mathrm{sub}}_{2\rho^{\mathrm{M}}}$ over X^{tor} (defined by projective and smooth cone decompositions) is ample, but subcanonical extensions are not preserved by tensor powers, and what can indeed be shown to be ample are more complicated. By Tai's work (see [AMRT75, Ch. IV, Sec. 2]) and the observation first made in [LS11, property (5) preceding (2.1)] (see also [LS13, Prop. 4.2(5)]), there exist an integer N_0 and a (possibly nonreduced) normal crossings divisor D' with $D'_{\mathrm{red}} = D$ such that $(\underline{W}^{\mathrm{can}}_{2\rho^{\mathrm{M}}})^{\otimes N}(-D')$ is ample for all $N \geq N_0$, and this turned out to be the most useful positivity property for our purpose. (As a consequence, although $\underline{W}^{\mathrm{can}}_{2\rho^{\mathrm{M}}}$ is generally not ample, it is still *nef and big.*)

Definition 3.1. *We say* $\nu \in \mathbf{X}^+_{\mathrm{M}}$ *is* **positive parallel** *if* $\dim_{\mathbb{C}} W_\nu = 1$ *and if the pullback of* ν *to each* $\mathbb{Q}$*-simple factor* G' *of* $\mathrm{G}_{\mathbb{C}}$ *is a positive rational multiple of* $\rho^{\mathrm{M}'}$*, for the corresponding factor* M' *of* M.

Example 3.2. Here are some examples of low ranks:

G	all positive parallel weights
SL_2	$\nu = k$ for $k \in \mathbb{Z}_{\geq 1}$
$\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SL}_2$, $F/\mathbb{Q}$ real quadratic	$\nu = k(1,1)$ for $k \in \mathbb{Z}_{\geq 1}$
$\mathrm{SL}_2 \times \mathrm{SL}_2$	$\nu = (k_1, k_2)$ for $k_1, k_2 \in \mathbb{Z}_{\geq 1}$
Sp_4	$\nu = k(1,1)$ for $k \in \mathbb{Z}_{\geq 1}$

Remark 3.3. See [Lan16c, Sec. 3] for a fairly complete description of the smallest positive parallel weights in all cases.

3.2. **Vanishing for coherent cohomology.**

Theorem 3.4 (see [LS12, Thm. 8.7 and 8.20], [LS13, Thm. 8.13 and 8.23], and [Lan16c, Thm. 4.1])**.** *Let $\nu \in X_{\mathrm{M}}^+$.*

(1) If there exists a positive parallel weight ν_- in X_{M}^+ such that $\nu + \nu_-$ is cohomological, then $H^i(X^{tor}, \underline{W}_\nu^{can}) = 0$ for $i < d - l(w(\nu + \nu_-))$.
(2) If there exists a positive parallel weight ν_+ in X_{M}^+ such that $\nu - \nu_+$ is cohomological, then $H^i(X^{tor}, \underline{W}_\nu^{sub}) = 0$ for $i > d - l(w(\nu - \nu_+))$.
(3) If there exist positive parallel weights ν_+ and ν_- in X_{M}^+ such that $\nu - \nu_+$ and $\nu + \nu_-$ are both cohomological, then the **interior cohomology**

$$H^i_{int}(X^{tor}, \underline{W}_\nu^{can}) := \mathrm{im}(H^i(X^{tor}, \underline{W}_\nu^{sub}) \to H^i(X^{tor}, \underline{W}_\nu^{can}))$$

satisfies $H^i_{int}(X^{tor}, \underline{W}_\nu^{can}) = 0$ for $i \notin [d - l(w(\nu + \nu_-)), d - l(w(\nu - \nu_+))]$.

Remark 3.5. Using the dual BGG decompositions (2.3) and (2.3) (which replace the usual Hodge decompositions), Theorem 3.4 can be viewed as a Kodaira-type vanishing theorem for the coherent cohomology of automorphic bundles, although (in noncompact cases) it did not follow from vanishing results readily available in the literature of algebraic geometry. (See [LS13, Sec. 1–3] and [Suh].)

3.3. **Vanishing for de Rham cohomology.**

Theorem 3.6 (see [LS12, Thm. 8.16], [LS13, Thm. 8.18], and [Lan16c, Thm. 4.10])**.** *Suppose $\mu \in \mathbf{X}_{\mathrm{G}_\mathbb{C}}^+$ is* **sufficiently regular** *in the sense that, for each $\alpha \in \Phi_{\mathrm{G}_\mathbb{C}}^+$, which comes from some $\mathbb{C}$-simple factor of $\mathrm{G}_\mathbb{C}$, we have:*

$$\langle \mu, \alpha^\vee \rangle \geq \begin{cases} 0, & \textit{if the factor is compact (i.e., roots are all in } \Phi_{\mathrm{M}}\textit{);} \\ 1, & \textit{if the factor is not compact and not of types } \mathrm{B} \textit{ or } \mathrm{C}\textit{;} \\ 2, & \textit{if the factor is not compact but is of types } \mathrm{B} \textit{ or } \mathrm{C}. \end{cases}$$

Then:

(1) $H^i_{dR}(X, \underline{V}_\mu^\vee) = 0$ for $i < d$.
(2) $H^i_{dR,c}(X, \underline{V}_\mu^\vee) = 0$ for $i > d$.
(3) The **interior cohomology**

$$H^i_{dR,int}(X, \underline{V}_\mu^\vee) := \mathrm{im}(H^i_{dR,c}(X, \underline{V}_\mu^\vee) \to H^i_{dR}(X, \underline{V}_\mu^\vee))$$

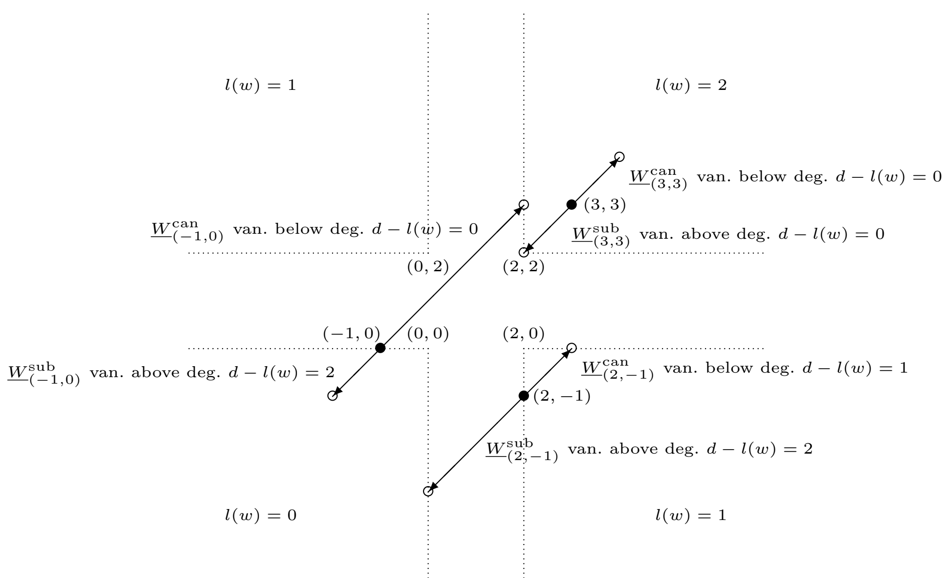
satisfies $H^i_{dR,int}(X, \underline{V}_\mu^\vee) = 0$ for $i \neq d$.

Remark 3.7. This follows from the vanishing for coherent cohomology (using the dual BGG decompositions (2.3) and (2.3)), and (in noncompact cases) reproves many Hermitian cases of Li and Schwermer's result (see [LS04, Cor. 5.6]). (See

[Lan16c, Rem. 4.16] for a more complete documentation of what were known earlier.) It is new (and beyond methods involving either automorphic or Galois representations) when Γ is not a congruence subgroup, although our understanding of the cohomology of automorphic bundles for such general Γ is still very limited.

3.4. **Some examples.**

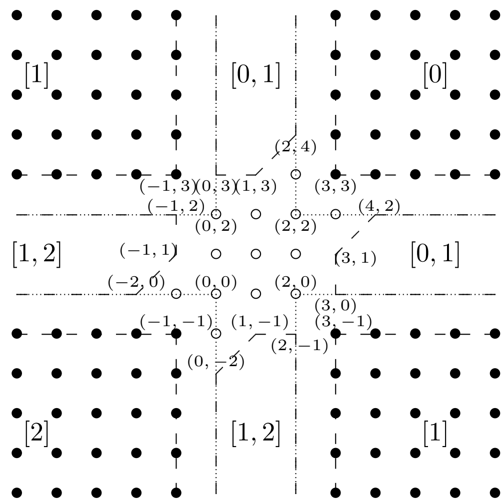
Example 3.8. This is an example for Theorem 3.4, which is about vanishing for the coherent cohomology, when $\mathrm{G} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SL}_2$ with $F/\mathbb{Q}$ real quadratic (which is the case of Hilbert modular surfaces). In the following diagram, starting with each weight represented by a bullet $\bullet$, we shift it by both positive and negative multiples of $(1,1)$ (which is the smallest positive parallel weight, as we have seen in Example 3.2), and consider the first cohomological weights we encounter in both directions, represented by two circles $\circ$. Then we record the Weyl lengths $l(w)$ associated with such cohomological weights, and determine the bounds $d - l(w)$ for vanishing degrees (with $d = 2$). For example, if we start with the weight $(2,-1)$, then the first cohomological weight we encounter after shifting by a positive integral multiple of $(1,1)$ is $(3,0)$, which is of Weyl length $l(w) = 1$, and hence we obtain the vanishing of the cohomology of $\underline{W}^{\mathrm{can}}_{(2,-1)}$ over X^{tor} below degree $d - l(w) = 1$. On the other hand, the first cohomological weight we encounter after shifting by a negative integral multiple of $(1,1)$ is $(0,-3)$, which is of Weyl length $l(w) = 0$, and hence we obtain the vanishing of the cohomology of $\underline{W}^{\mathrm{sub}}_{(2,-1)}$ over X^{tor} above degree $d - l(w) = 2$ (which is useless because the cohomology of a surface always vanishes above degree $d = 2$). The cases for the weights $(3,3)$ and $(-1,0)$ are similar.



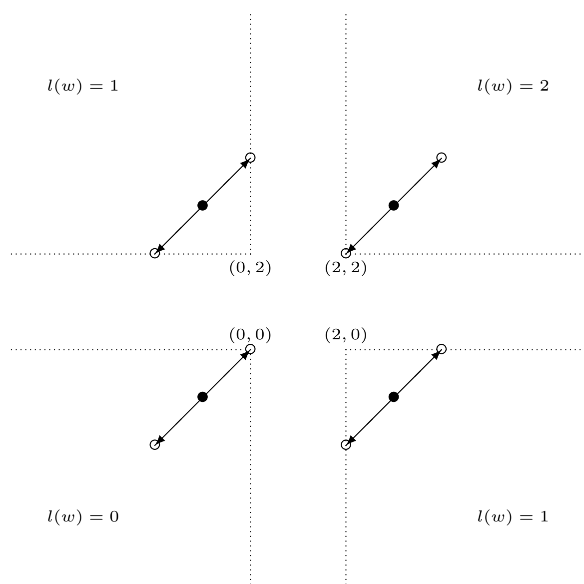
By considering all weights, we obtain the following summarizing diagram (see [Lan16c, Ex. 4.18] for more details), in which an interval $[a, b]$ (or simply $[a] = [a, a]$) means $H^i(X^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_\nu) = 0$ for $i < a$, $H^i(X^{\mathrm{tor}}, \underline{W}^{\mathrm{sub}}_\nu) = 0$ for $i > b$, and

$H^i_{\mathrm{int}}(X^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_\nu) = 0$ for $i \notin [a, b]$:

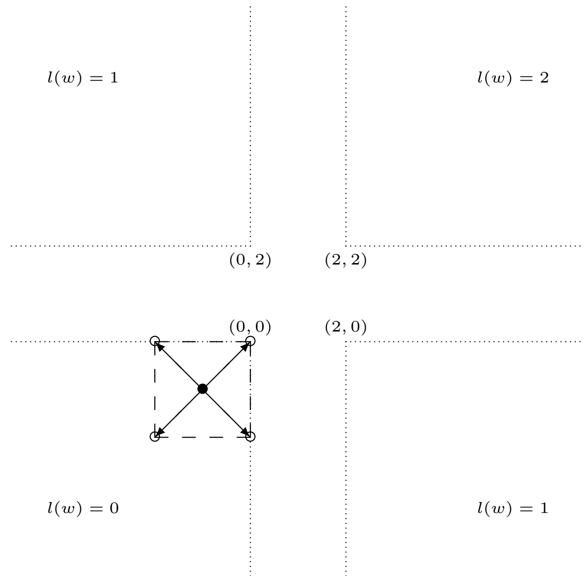
•: regular and concentrate in one degree
○: [0, 2] (useless)



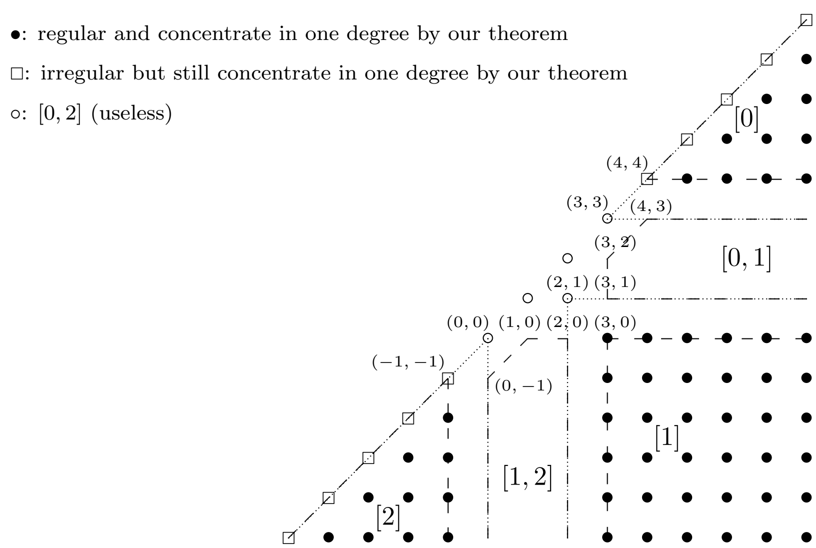
Example 3.9. This is an example for Theorem 3.4, which is about vanishing for the de Rham cohomology, again when $\mathrm{G} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SL}_2$ with $F/\mathbb{Q}$ real quadratic. In the following diagram, the bullets • denote the weights of the dual BGG pieces of the sufficiently regular weight $\mu = (1, 1)$, and the upshot is that these weights remain cohomological (with the same associated Weyl length) when shifted by the smallest positive parallel weight $(1, 1)$ and the opposite $(-1, -1)$.



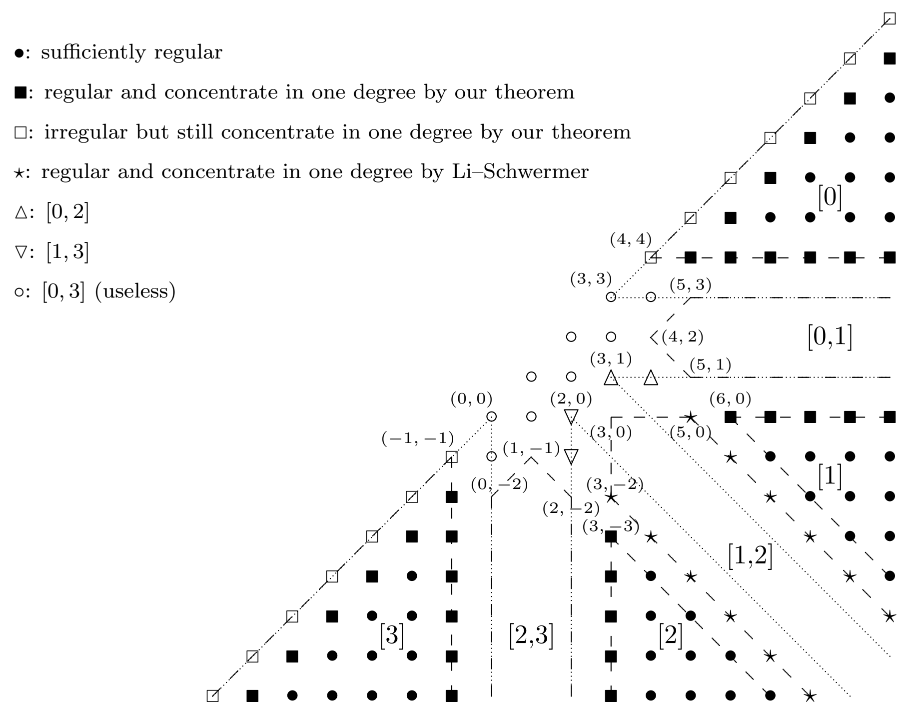
In the following diagram, we apply the Weyl group action and move all weights to the cohomological region of Weyl length zero. Thus, the point of being sufficiently regular is that the weight then remains (up to duality) in the dominant chamber after shifting by the whole Weyl orbit of the smallest positive parallel weight.



Example 3.10. This is an example for Theorem 3.4 when $\operatorname{Lie} \mathrm{G}_{\mathbb{R}} \cong \mathfrak{su}_{2,1}$ (which is the case of Picard modular surfaces; see [Lan16c, Ex. 4.19] for more details). (In the following diagram, we visualize $(k_1, k_2; k_3)$ mod $(1, 1; 1)$ by $(k_1 - k_3, k_2 - k_3)$.)



Example 3.11. This is an example for Theorem 3.4 when $\mathrm{G} = \mathrm{Sp}_4$ (which is the case of Siegel threefolds; see [Lan16c, Ex. 4.17] for more details).



Example 3.12. This is an example for Theorem 3.4 when $\operatorname{Lie} \mathrm{G}_{\mathbb{R}} \cong \mathfrak{e}_{7(-25)}$. In this case, we embed $\Phi_{\mathrm{G}_{\mathbb{C}}}$ in $\mathbb{R}^7$ (with Killing form induced, up to scaling, by the Euclidean inner product) with positive simple roots in $\Phi^+_{\mathrm{G}_{\mathbb{C}}}$ given by

$$\alpha_1 = (1, -1, 0, 0, 0, 0, 0), \quad \alpha_2 = (0, 1, -1, 0, 0, 0, 0),$$
$$\alpha_3 = (0, 0, 1, -1, 0, 0, 0), \quad \alpha_4 = (0, 0, 0, 1, -1, 0, 0),$$
$$\alpha_5 = (0, 0, 0, 0, 1, -1, 0), \quad \alpha_6 = (0, 0, 0, 0, 1, 1, 0),$$
$$\alpha_7 = (-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{\sqrt{2}}{2}),$$

with $\alpha_1 \notin \Phi_{\mathrm{M}}$. Then we have (see [Lan16c, Ex. 5.46, 5.47, and 5.48]):

ν	cohomological?	$\mu(\nu)$ regular?	$H^i_{\mathrm{int}} = 0$ for?
$(10, 10, 9, 7, 4, 0, 26\sqrt{2})$	yes	yes	$i \neq 6$
$(-14, 8, 3, 2, 1, 0, \sqrt{2})$	yes	no	$i \notin [25, 27]$
$(-7, 5, 5, 2, 1, 0, 3\sqrt{2})$	no	undefined	$i \notin [23, 24]$

3.5. **Relative vanishing.** Let $\pi : X^{\mathrm{tor}} \to X^{\mathrm{min}}$ denote the canonical morphism. A rather unpredicted recent discovery about the cohomology of automorphic bundles is the following:

Theorem 3.13 (see [LS14] and [Lan16c, Thm. 4.5])**.** *For every $\nu \in \mathbf{X}^+_{\mathrm{M}}$, we have $R^i \pi_* \underline{W}^{sub}_{\nu} = 0$ for all $i > 0$.*

Remark 3.14. It is not true in general that $R^i \pi_* \underline{W}^{\mathrm{can}}_{\nu} = 0$ for all $i > 0$.

Remark 3.15. The first proofs of results like Theorem 3.13 (see, for example, [HLTT16, Thm. 5.4], [Lan18a, Sec. 8.2], and [Lan16b, Thm. 3.9 and Rem. 10.1]) were based on detailed analyses of the formal fibers of π. The later (shorter) proofs in [LS14] and [Lan16c, Thm. 4.5] were based on (various versions of) Theorem 3.4.

3.6. **Higher Koecher's principle.** The relative vanishing of Theorem 3.13 implies the following generalization of the classical Koecher's principle, whose discovery was surprisingly late, given how fundamental the assertion seems to be:

Theorem 3.16 (higher Koecher's principle; see [Lan16b, Thm. 2.5] and [Lan16c, Thm. 4.7])**.** *Let* $\nu \in \mathbf{X}^+_{\mathrm{M}}$. *Let*

$$c_X := \mathrm{codim}_{\mathbb{C}}(X^{min} - X, X^{min}).$$

Let us denote by $j^{tor} : X \hookrightarrow X^{tor}$ *and* $j^{min} : X \hookrightarrow X^{min}$ *the canonical morphisms. Then the canonical morphism* (*induced by* j^{tor})

$$R^i\pi_*\underline{W}^{can}_\nu \to R^i j^{min}_* \underline{W}_\nu$$

is an isomorphism for $i < c_X - 1$, *and is injective for* $i = c_X - 1$. *Hence, by the Leray spectral sequence, for any open* U *in* X^{min}, *the canonical restriction*

$$H^i(\pi^{-1}(U), \underline{W}^{can}_\nu) \to H^i((j^{min})^{-1}(U), \underline{W}_\nu)$$

is bijective (*resp. injective*) *for all* $i < c_X - 1$ (*resp.* $i = c_X - 1$).

Remark 3.17. This theorem implies its complex analytic analogue, by the same argument as in [Lan16b, Sec. 3], using GAGA (see [Ser56] and [Gro71, XII]) and the results on local cohomology in [Gro68, VIII] and [Siu70].

Remark 3.18. When $i = 0$, $U = X^{\mathrm{min}}$, and $c_X > 1$, this specializes to the classical Koecher's principle, which asserts that a section of $\underline{W}_\nu$ over $(j^{\mathrm{min}})^{-1}(U) = X$ automatically extends to a section of $\underline{W}^{\mathrm{can}}_\nu$ over all of $\pi^{-1}(U) = X^{\mathrm{tor}}$, or in other words that the *growth condition* is unnecessary in the definition of such (generalizations of) modular forms. (See [Lan16b, Sec. 2].)

Example 3.19. Applying Theorem 3.16 with $U = X^{\mathrm{min}}$, we see that the canonical restriction map

$$H^i(X^{\mathrm{tor}}, \underline{W}^{\mathrm{can}}_\nu) \to H^i(X, \underline{W}_\nu) \tag{3.20}$$

is bijective if $i < c_X - 1$, and injective if $i = c_X - 1$. For the values of c_X, we have the following summarizing table:

G (most split form)	$\mathrm{rk}_{\mathbb{Q}}$	$\dim_{\mathbb{C}}(X)$	$\dim_{\mathbb{C}}(X^{\mathrm{min}} - X)$	c_X
SL_2	1	1	0	1
$\mathrm{SL}_{2,F}$, $F/\mathbb{Q}$ totally real	1	$d = [F : \mathbb{Q}]$	0	d
Sp_4	2	3	1	2
Sp_{2n}, $n \geq 1$	n	$\frac{1}{2}n(n+1)$	$\frac{1}{2}n(n-1)$	n
$\mathrm{Lie}\,\mathrm{G}_{\mathbb{R}} \cong \mathfrak{su}_{2,1}$	1	2	0	2
$\mathrm{Lie}\,\mathrm{G}_{\mathbb{R}} \cong \mathfrak{su}_{n,1}$, $n \geq 1$	1	n	0	n
$\mathrm{Lie}\,\mathrm{G}_{\mathbb{R}} \cong \mathfrak{su}_{a,b}$, $ab \geq 1$	$\min(a, b)$	ab	$(a-1)(b-1)$	$a+b-1$
$\mathrm{Lie}\,\mathrm{G}_{\mathbb{R}} \cong \mathfrak{so}_{1,2}$	1	1	0	1
$\mathrm{Lie}\,\mathrm{G}_{\mathbb{R}} \cong \mathfrak{so}_{n,2}$, $n \geq 2$	2	n	1	$n-1$
$\mathrm{Lie}\,\mathrm{G}_{\mathbb{R}} \cong \mathfrak{so}^*_{2n}$, $n \geq 2$	$\lfloor \frac{n}{2} \rfloor$	$\frac{1}{2}n(n-1)$	$\frac{1}{2}(n-2)(n-3)$	$2n-3$
$\mathrm{Lie}\,\mathrm{G}_{\mathbb{R}} \cong \mathfrak{e}_{6(-14)}$	2	16	5	11
$\mathrm{Lie}\,\mathrm{G}_{\mathbb{R}} \cong \mathfrak{e}_{7(-25)}$	3	27	10	17

Remark 3.21. If $i = c_X - 1$, then (3.20) can actually fail to be surjective, up to shifting ν by a sufficiently large multiple of $2\rho^{\mathrm{M}}$ (by the same argument as in [Lan16b, Sec. 9]). This suggests that there are some mysterious *fake modular forms* in degree $c_X - 1$. But we have no idea what they are—the proof of their existence is based on [Gro68, VIII, Prop. 3.2], a general fact in the theory of local cohomology.

3.7. **Vanishing over integral models.** To prove analogues of the above theorems with coefficients over integers or their reductions modulo integers, we can no longer just work with the complex analytic locally symmetric varieties—we need certain good integral models. (We are being vague here about what "good" means.)

So far all known constructions of good integral models rely on an important coincidence: When $\mathrm{G} = \mathrm{Sp}_{2n}$ or better GSp_{2n} (to avoid introducing roots of unity into the base rings), X is a moduli space of principally polarized abelian varieties with some level structures over $\mathbb{C}$. This led to the construction of good integral models of X for the *PEL-* and *Hodge-type* Shimura varieties, by taking normalizations over moduli of *abelian schemes* with PEL structures (i.e., polarizations, endomorphism structures, and level structures) and with certain Hodge tensors. (See, for example, [Kot92] and [Lan13] for the construction of *smooth PEL moduli*; see [Kis10] and [KMP16] for the construction of good reduction integral models of Hodge-type Shimura varieties; and see [Lan16a] for the construction by normalization in all PEL-type cases, allowing bad reduction.) We now know how to also construct good integral models of X^{tor} and X^{min} in such cases, allowing arbitrary ramifications, levels, polarization degrees, and isogeny collections in all PEL-type cases. (See [Lan13], [Lan12c], [Lan12a], [Lan16a], [Lan15], [Lan17], and [Lan18b] for various constructions and comparisons in PEL-type cases; and see [MP15] for a different approach in Hodge-type cases, with some restriction on the levels at p allowing all hyperspecial and many parahoric ones.) The construction of good reduction integral models of X extends to the *abelian-type* cases—see [Kis10] and [KMP16] again. In [KP15], the construction is further extended to allow bad reductions with parahoric levels at p (under the assumptions that $p > 2$ and that G splits over a tamely ramified extension of $\mathbb{Q}_p$). (The construction for X^{tor} and X^{min} should also extend to abelian-type cases with parahoric levels at p, but is not carried out yet.)

At least in PEL-type cases, the automorphic bundles $\underline{W}_\nu$ and their extensions $\underline{W}_\nu^{\mathrm{can}}$ and $\underline{W}_\nu^{\mathrm{sub}}$ can also be defined over even bad reduction integral models of X. In good reduction cases, we can also consider $(\underline{V}_\mu, \nabla)$ over smooth integral models of X, their extensions $(\underline{V}_\mu^{\mathrm{can}}, \nabla)$ and $(\underline{V}_\mu^{\mathrm{sub}}, \nabla)$ over smooth integral models X^{tor} with log poles over $D = (X^{\mathrm{tor}} - X)_{\mathrm{red}}$, and their (log) de Rham cohomology; and we have p-torsion or p-integral analogues of Theorems 3.4 and 3.6 when p is unramified and larger than a bound (dependent on ν or μ but) independent of the level away from p. The short exact sequence

$$0 \to \underline{W}_{\mathbb{Z}_{(p)}}^{\mathrm{can}} \xrightarrow{p} \underline{W}_{\mathbb{Z}_{(p)}}^{\mathrm{can}} \to \underline{W}_{\mathbb{Z}/p\mathbb{Z}}^{\mathrm{can}} \to 0$$

induces a long exact sequence

$$\cdots \to H^{i-1}(\underline{W}_{\mathbb{Z}/p\mathbb{Z}}^{\mathrm{can}}) \to H^i(\underline{W}_{\mathbb{Z}_{(p)}}^{\mathrm{can}}) \xrightarrow{p} H^i(\underline{W}_{\mathbb{Z}_{(p)}}^{\mathrm{can}}) \to H^i(\underline{W}_{\mathbb{Z}/p\mathbb{Z}}^{\mathrm{can}}) \to \cdots .$$

Hence, vanishing mod p induces not just vanishing over $\mathbb{Z}_{(p)}$, but also *freeness* and *liftability*. By *crystalline comparison* (see [LS12, Sec. 5] and [LS13, Sec. 9], and the references there), under certain assumptions on p (and on the local properties of the integral models), the vanishing of (log) de Rham cohomology mod p implies the vanishing of the corresponding étale and Betti cohomology with coefficients mod p (which then also implies the corresponding freeness and liftability assertions). (For more details, see [LS12], [LS13], and [LP]; see also the survey article [Lan12b].)

On the other hand, the relative vanishing in Theorem 3.13 has analogues over integral models of PEL-type Shimura varieties even in bad reduction cases (see, for example, [Lan17, Thm. 8.6] and [Lan18b, Thm. 4.4.9]), which are important because, roughly speaking, many new techniques for congruences involve the consideration of the sheaves $\pi_* \underline{W}_\nu^{\mathrm{sub}}$ over $X^{\min}$. (There is no restriction at all on the residue characteristics involved. We do not need them to be larger than any bounds dependent on, for example, the levels or weights.) However, in such generality, we have to resort to some detailed analyses of formal fibers of π, because we no longer expect any Kodaira-type vanishing as in Theorem 3.4 to be true (cf. Remark 3.15). Thus, the nature of the relative vanishing as in Theorem 3.13 remains mysterious.

4. Application to the construction of Galois representations

4.1. **Conjectural framework and historical developments.** Let us begin with the following conjectural framework (incorporating conjectures due to Langlands, Clozel [Clo90], Fontaine–Mazur [FM97], and some others). Given any prime number p, any field isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, any number field F, and any $n \in \mathbb{Z}_{\geq 1}$, it is conjectured that there is a natural bijection:

$$\left\{ \begin{matrix} \text{irreducible algebraic cuspidal} \\ \text{automorphic representations} \\ \pi \cong \otimes'_v \pi_v \text{ of } \mathrm{GL}_n(\mathbb{A}_F) \end{matrix} \right\}_{/\cong} \leftrightarrow \left\{ \begin{matrix} \text{irreducible algebraic} \\ \text{continuous representations} \\ r : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p) \end{matrix} \right\}_{/\cong}$$

The term *algebraic* at the left-hand side means the Harish-Chandra parameter of the archimedean component $\pi_\infty = \otimes_{v|\infty} \pi_v$ is integral after shifting by the half-sum of positive roots. The term *algebraic* at the right-hand side means r is unramified at all but finitely many places and de Rham at all $v|p$. (These are often called *geometric* because they are satisfied by the p-adic étale cohomology of varieties over F.) The bijection is *natural* in the sense that it matches the local L-factors at both sides—or it suffices to know the *local-global compatibility* at all finite places $v \nmid p$ over which F, π_v, and r_v are all unramified, by matching (under ι) the Satake parameters of π_v with the Frobenius eigenvalues of r_v (up to some normalization).

Here is a summary of some historical developments:

- When $n = 1$, the conjecture is known, thanks to *class field theory*.
- When $n > 1$, the conjecture is often studied in two directions: *attaching Galois representations* r to automorphic representations π, and the converse (i.e., *modularity* or *automorphy* problems).
- When F is CM or totally real, for cohomological (i.e., regular algebraic) and *polarized* π, one can construct r by traditional methods: by realizing r (up to base change, patching, and twisting) in the étale cohomology of *Shimura varieties* when π_∞ has sufficient regular weights (which is often proved by trace formula techniques), and by extending such r to other cohomological weights by congruences. (One can further extend the construction to cases where π_∞ is some holomorphic limits of discrete series, realized in H^0, by congruences; see, for example, [Tay91].)

- The modularity or automorphy problems are often studied by the so-called *Taylor–Wiles method*. For the method to work in higher dimensions, one often needs the cohomology to behave like free modules over some auxiliary Hecke algebras, and this requires rather strong vanishing results.

4.2. **Removal of polarizability condition.**

Theorem 4.1 (Harris–Lan–Taylor–Thorne; see [HLTT16, Introduction, Thm. A])**.** *Let* p, $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$, *and* n *be as above, and* F *totally real or CM. Suppose* π *is a* **cohomological** *cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_F)$ (*which is then necessarily algebraic*). *Then there is a unique semisimple representation*

$$r = r_{p,\iota}(\pi) : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

such that, if $\ell \neq p$ *is a prime above which both* F *and* π *are unramified and if* $v|\ell$ *is a place of* F, *then* r *is unramified at* v *and satisfies the local-global compatibility*

$$r|^{ss}_{W_{F_v}} = \iota^{-1} \operatorname{rec}_{F_v}(\pi_v |\det|_v^{\frac{1-n}{2}}).$$

Remark 4.2. The most important point of Theorem 4.1 is that we impose *no polarizability condition* on π (i.e., not requiring π to be conjugate self-dual up to character twists). Then r generally does not occur in the p-adic étale cohomology of any Shimura variety!

Instead, we constructed r using *p-adic limits* of Galois representations which do occur in the p-adic étale cohomology of some Shimura varieties. The main ideas can be briefly (and somewhat imprecisely) summarized as follows:

(1) We may assume that $n > 1$. Also, by patching, we may assume that F is CM and satisfies certain simplifying assumptions.
(2) The starting point is Skinner's idea that, to construct $r = r_{p,\iota}(\pi)$, it suffices to construct $R_t = r_{p,\iota}(\pi|\det|^t) \oplus r_{p,\iota}(\pi|\det|^t)^{c,\vee}$ for sufficiently many t.
(3) Each R_t should correspond to Eisenstein series for "$\mathrm{GU}(n, n)$" (with Levi $\mathrm{GL}_{n,F} \times \mathrm{G_m}$). But nothing along that line really works.
(4) Rather, we construct them as *overconvergent cusp forms*, which are global sections of $\pi_* \underline{W}_\nu^{\mathrm{sub},\dagger}$ (i.e., the overconvergent version of $\pi_* \underline{W}_\nu^{\mathrm{sub}}$) over the affinoid (multiplicative-type) ordinary locus $X^{\mathrm{ord,min},\dagger}$ of $X^{\mathrm{min},\dagger}$ for some $\nu \in \mathbf{X}^+_{\mathrm{M}}$, for some "$\mathrm{GU}(n, n)$" Shimura varieties X. (Here the superscripts $^{\mathrm{ord}}$ and † mean the *dagger spaces* attached to the tubes defined by certain p-integral models with only *ordinary loci* in characteristic p.) They are naturally p-adic limits of cusp forms whose associated Galois representations were already known (thanks to historical developments in the *polarizable case*, because we are working on Shimura varieties associated with unitary similitude groups over CM fields).
(5) Using the mysterious relative vanishing $R^i \pi_* \underline{W}_\nu^{\mathrm{sub},\dagger} = 0$ for $i > 0$, it suffices to construct sections of $\underline{W}_\nu^{\mathrm{sub},\dagger}$ over the (non-affinoid) ordinary locus $X^{\mathrm{ord,tor},\dagger}$ of $X^{\mathrm{tor},\dagger}$, and we achieved this using the Hodge spectral sequence of certain *rigid cohomology of* $\overline{X}^{ord}$ *with compact support along the partial boundary* $\overline{X}^{ord,tor} - \overline{X}^{ord}$, or of the analogue with X and X^{tor} replaced with certain Kuga families (which are self-fiber-products of the universal abelian schemes over the original X) and their toroidal compactifications. (Here the overlines and the superscripts $^{\mathrm{ord}}$ mean the characteristic p fibers

of certain integral models with only ordinary loci, the same ones used in the definition of the dagger spaces above.)

(6) The key observation is the following: The partial boundary $\overline{X}^{\mathrm{ord,tor}} - \overline{X}^{\mathrm{ord}}$ and its analogue for Kuga families can be arranged to be simple normal crossings divisors such that the incidence relations of their smooth irreducible components are essentially the same as in characteristic zero, which encode the interior cohomology of the (real analytic) locally symmetric manifold associated with $\mathrm{GL}_{n,F}$ (with coefficients in local systems associated with finite-dimensional representations of $\mathrm{GL}_{n,F}$ of polynomial weights). Then a Frobenius-weight argument (based on [Ber97, Thm. 3.1] and [Chi98, Thm. 2.2]) shows that such interior cohomology contributes to the rigid cohomology mentioned above. (Note that we are really realizing the interior cohomology of a real analytic manifold in the rigid cohomology of some characteristic p algebraic variety.) But all π considered in Theorem 4.1 (or more precisely their nonarchimedean components) contribute up to determinant twists (and ι) to such interior cohomology!

Remark 4.3. In [Sch15], Scholze reproved Theorem 4.1 using a much more advanced method, and also treated the analogue for torsion cohomological classes—in fact, the consideration of torsion classes is crucial in his argument. Nevertheless, to show that the Galois representation r constructed in Theorem 4.1 is indeed *algebraic*, at least with current techniques, it still seems easier to use the results in [HLTT16].

There have been many other exciting developments since [HLTT16] and [Sch15], but it is not easy to give a fair overview of all of them, and we decided to stop here due to limitation of time and energy.

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School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA