LIFTABILITY OF MOD p CUSP FORMS OF PARALLEL WEIGHTS

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Abstract. We present a vanishing theorem for automorphic line bundles on good reduction fibers of PEL-type Shimura varieties (including all noncompact ones). As a consequence, we deduce that for a good prime p no smaller than the dimension of a PEL-type Shimura variety, any mod p cusp form of positive cohomological parallel weight is liftable to characteristic zero.

1. Introduction

In the development of a geometric theory of p-adic modular forms, a question of fundamental interest is whether classical modular forms defined geometrically over a finite field can be *lifted*, that is, are reductions mod p of forms similarly defined in characteristic zero. One could naïvely hope this is always possible, but there are counterexamples in low dimensions (see Remarks 4.5 and 4.6). The aim of this article is to provide an affirmative answer under some mild and effective conditions for *cusp forms* defined by automorphic *line bundles* on (possibly noncompact) PEL-type Shimura varieties.

The main results of this paper can be summarized as follows:

Theorem 1.1 (Theorem 4.1 and Corollary 4.3)**.** *On a good reduction fiber of a PEL-type Shimura variety of dimension no greater than the residue characteristic p, all the higher cohomology of an automorphic line bundle of positive cohomological parallel weight vanishes. Consequently, any mod p cusp form of such a weight is liftable to characteristic zero.*

The precise definition of the Shimura varieties and the automorphic line bundles will be given in Section 2.

There are two main ingredients in our proof. The first is Esnault and Viehweg's vanishing theorem in positive characteristic (see Section 3). The second is the theory of toroidal compactifications over mixed characteristic bases, developed in [6] and [8], which allows us to verify the two crucial conditions (of liftability and positivity) in Esnault and Viehweg's theorem.

Although the proofs are short and simple, we emphasize that all conditions in our statements are effective and independent of the level. After this article was submitted, we learned that B. Stroh [15] recently obtained similar results in the Siegel case for genus 2 or 3, also using the vanishing theorem of Esnault and Viehweg. His method depends on interesting positivity properties of automorphic line bundles that are, however, peculiar to the Siegel case [14]. We note that in the

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Siegel case (for any genus), our method depends only on a result of Faltings and Chai [6, Ch. V, Thm. 5.8], and neither on [14] nor on the first author's thesis.

We shall follow [8, Notations and Conventions] unless otherwise specified.

2. Geometric Setup

2.1. **PEL-Type Moduli Problems.** Let $(\mathcal{O}, \star, L, \langle\cdot,\cdot\rangle, h_0)$ be an integral PEL datum in the following sense:

(1) $\mathcal{O}$ is an order in a (nonzero) simple algebra, finite-dimensional over $\mathbb{Q}$, with a positive involution $\star$.
(2) $(L, \langle\cdot,\cdot\rangle, h_0)$ is a PEL-type $\mathcal{O}$-lattice as in [8, Def. 1.2.1.3] (in [8, Def. 1.2.1.3] h_0 was denoted by h).

Any such datum defines (by [8, Def. 1.2.1.5] and [8, Def. 1.2.5.4]) a group functor G over $\mathrm{Spec}(\mathbb{Z})$ and a number field F_0 called the reflex field.

We shall denote the ring of integers in F_0 by $\mathcal{O}_{F_0}$ and use similar notations for other number fields. This is in conflict with the notation of the order $\mathcal{O}$ in the integral PEL datum, but the precise interpretation will be clear from the context.

We say that a rational prime number $p > 0$ is *good* if it satisfies the following conditions (cf. [8, Def. 1.4.1.1]):

(1) p is unramified in $\mathcal{O}$ (as in [8, Def. 1.1.1.14]).
(2) $p \neq 2$ if $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ involves simple factors of type D (as in [8, Def. 1.2.1.15]).
(3) The pairing $\langle\cdot,\cdot\rangle$ is perfect after base change to $\mathbb{Z}_{(p)}$.

Let us choose a good prime p, which will be fixed throughout, and let $\mathcal{H}$ be a *neat* open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^p)$ (see [13, 0.6] or cf. [8, Def. 1.4.1.8] for the definition of neatness).

By [8, Def. 1.4.1.4] (with $\Box = \{p\}$ there), the data of $(L, \langle\cdot,\cdot\rangle, h_0)$ and $\mathcal{H}$ define a moduli problem $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, parameterizing tuples $(A, \lambda, i, \alpha_{\mathcal{H}})$ over schemes S over S_0 of the following form:

(1) $A \to S$ is an abelian scheme, and $\lambda : A \to A^\vee$ is a polarization of degree prime to p.
(2) $i : \mathcal{O} \hookrightarrow \mathrm{End}_S(A)$ is an $\mathcal{O}$-endomorphism structure as in [8, Def. 1.3.3.1].
(3) $\underline{\mathrm{Lie}}_{A/S}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}$-module structure given naturally by i satisfies the determinantal condition in [8, Def. 1.3.4.2] given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\cdot,\cdot\rangle, h_0)$.
(4) $\alpha_{\mathcal{H}}$ is an (integral) level-$\mathcal{H}$ structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p, \langle\cdot,\cdot\rangle)$ as in [8, Def. 1.3.7.8].

(The definition can be identified with the one in [7, §5] by [8, Prop. 1.4.3.3].) By [8, Thm. 1.4.1.12 and Cor. 7.2.3.10], $\mathsf{M}_{\mathcal{H}}$ is representable by a (smooth) quasi-projective scheme over S_0 (under the assumption that $\mathcal{H}$ is neat).

According to [8, Thm. 6.4.1.1 and 7.3.3.4], under the assumption that $\mathcal{H}$ is neat, $\mathsf{M}_{\mathcal{H}}$ admits a *toroidal compactification* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} = \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$, a scheme projective and smooth over S_0, depending on a compatible collection Σ (of the so-called cone decompositions) which is *projective* and *smooth* in the sense of [8, Def. 6.3.3.2 and 7.3.1.3]. It satisfies the following properties:

(1) Let d be the relative dimension of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ over S_0. This number can be effectively calculated from the PEL datum we started with (see [8, Cor.

2.2.4.15 and Thm. 2.2.4.16].) and agrees with the complex dimension of the finite union $\mathsf{X} = \mathrm{G}(\mathbb{R})h_0$ of Hermitian symmetric spaces.

(2) The universal abelian scheme $A \to \mathsf{M}_{\mathcal{H}}$ extends to a semi-abelian scheme $A^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$. In particular, we can define the relative cotangent bundle $\underline{\mathrm{Lie}}^{\vee}_{A^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} := e^*_{A^{\mathrm{tor}}}\Omega^1_{A^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$, which is locally free and coherent.

(3) The complement $(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} - \mathsf{M}_{\mathcal{H}})_{\mathrm{red}}$ is a relative Cartier divisor $\mathsf{D} = \mathsf{D}_{\infty,\mathcal{H}}$ with *simple* normal crossings. Here simpleness of the normal crossings uses [8, Cond. 6.2.5.18 and Lem. 6.2.5.20] and the assumption that $\mathcal{H}$ is neat.

(4) Let $\omega := \wedge^{\mathrm{top}}\,\underline{\mathrm{Lie}}^{\vee}_{A^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$. Then, according to [8, Thm. 7.2.4.1], the scheme $\mathrm{Proj}(\underset{r\geq 0}{\oplus}\Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}},\omega^{\otimes r}))$ over S_0 is projective and normal, containing $\mathsf{M}_{\mathcal{H}}$ as an open dense subscheme, and defines the *minimal compactification* $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ of $\mathsf{M}_{\mathcal{H}}$ (independent of the choice of Σ). Moreover, the invertible sheaf ω descends to an ample invertible sheaf over $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$.

(5) Under the assumption that Σ is projective, [8, Thm. 7.3.3.4] asserts more precisely that $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is the normalization of the blowup of $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ along a coherent sheaf of ideals $\mathcal{J}$ of $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}}$ whose pullback $\jmath$ to $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ is of the form $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}(-\mathsf{D}')$, for some relative Cartier divisor D' of normal crossings on $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ such that $\mathsf{D}'_{\mathrm{red}} = \mathsf{D}$. In particular,

$$\exists\, r_0 > 0 \text{ such that } \omega^{\otimes r}(-\mathsf{D}') \text{ is ample for every } r \geq r_0. \tag{2.1}$$

(All these assertions are generalizations of their counterparts in the Siegel case [6].) In what follows, we shall fix a choice of Σ, and suppress Σ from the notations.

Because $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ is simple, its center F is a number field . By [8, Prop. 1.1.1.17], we have a non-canonical isomorphism $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_p \cong \mathrm{M}_t(\mathcal{O}_F\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_p)$ for some integer $t > 0$.

Let R_1 be any discrete valuation ring over $\mathcal{O}_{F_0,(p)}$ satisfying the following conditions:

(1) The maximal ideal of R_1 is generated by p, and the residue field κ_1 of R_1 is a perfect field of characteristic p. In this case, the p-adic completion of R_1 is isomorphic to the Witt vectors $W(\kappa_1)$ over κ_1.

(2) $\mathcal{O}\underset{\mathbb{Z}}{\otimes}R_1 \cong \mathrm{M}_t(\mathcal{O}_F\underset{\mathbb{Z}}{\otimes}R_1)$.

Let $\mathsf{S}_1 := \mathrm{Spec}(R_1)$. Let $L_1 := (\mathcal{O}_F\underset{\mathbb{Z}}{\otimes}R_1)^{\oplus t}$ be (up to isomorphism) the unique simple projective (left) $\mathcal{O}\underset{\mathbb{Z}}{\otimes}R_1$-module of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}R_1 \cong \mathrm{M}_t(\mathcal{O}_F\underset{\mathbb{Z}}{\otimes}R_1)$.

2.2. **PEL-Type Shimura Varieties.** Consider the (real analytic) set $\mathsf{X} = \mathrm{G}(\mathbb{R})h_0$ of $\mathrm{G}(\mathbb{R})$-conjugates $h : \mathbb{C} \to \mathrm{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L\underset{\mathbb{Z}}{\otimes}\mathbb{R})$ of $h_0 : \mathbb{C} \to \mathrm{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L\underset{\mathbb{Z}}{\otimes}\mathbb{R})$. Let $H^p := \mathcal{H}$ and $H_p := \mathrm{G}(\mathbb{Z}_p)$ be open compact subgroups of $\mathrm{G}(\hat{\mathbb{Z}}^p)$ and $\mathrm{G}(\mathbb{Z}_p)$, and let H be the open compact subgroup H^pH_p of $\mathrm{G}(\hat{\mathbb{Z}})$. It is well known (see [7, §8] or [9, §2]) that there exists a quasi-projective variety Sh_H over F_0, together with a canonical open and closed immersion $\mathsf{Sh}_H \hookrightarrow \mathsf{M}_{\mathcal{H}}\underset{\mathcal{O}_{F_0,(p)}}{\otimes}F_0$ when $\mathcal{H}$ is neat, such that the analytification of $\mathsf{Sh}_H\underset{F_0}{\otimes}\mathbb{C}$ can be canonically identified with the double coset space $\mathrm{G}(\mathbb{Q})\backslash\mathsf{X}\times\mathrm{G}(\mathbb{A}^{\infty})/H$.

Let $\mathsf{M}_{\mathcal{H},0}$ (resp. $\mathsf{M}_{\mathcal{H},0}^{\mathrm{tor}}$, resp. $\mathsf{M}_{\mathcal{H},0}^{\mathrm{min}}$) denote the schematic closure of Sh_H in $\mathsf{M}_{\mathcal{H}}$ (resp. in $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, resp. in $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$). Then $\mathsf{M}_{\mathcal{H},0}$ is smooth over S_0, and $\mathsf{M}_{\mathcal{H},0}^{\mathrm{tor}}$ is proper smooth over S_0 with properties analogous to those of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$.

2.3. **Modular Forms of Cohomological Parallel Weights.** Let $\mathsf{M}_{\mathcal{H},1}$ (resp. $\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}$, resp. $\mathsf{M}_{\mathcal{H},1}^{\mathrm{min}}$) denote the pullback of $\mathsf{M}_{\mathcal{H},0}$ (resp. $\mathsf{M}_{\mathcal{H},0}^{\mathrm{tor}}$, resp. $\mathsf{M}_{\mathcal{H},0}^{\mathrm{min}}$) under $\mathsf{S}_1 \to \mathsf{S}_0$. For simplicity, we shall denote the pullbacks of A (resp. A^{tor}) to $\mathsf{M}_{\mathcal{H},1}$ (resp. $\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}$) by the same notation. When the context is clear, we shall also denote the pullbacks of other objects (such as D and D') by the same notations.

Lemma 2.1. *The sheaf* $\underline{\mathrm{Hom}}_{\mathcal{O}}(L_1, \underline{\mathrm{Lie}}^{\vee}_{A^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}})$ *is locally (over* $\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}$*) a free module of finite rank over* $\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}}$ *(and hence over* $\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}}$*).*

Proof. The sheaf $\underline{\mathrm{Lie}}^{\vee}_{A^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}}$ is locally (over $\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}$) a direct sum of finitely many copies of the sheaf $L_1 \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}}$. Hence the result follows from the fact that $\mathrm{Hom}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R_1}(L_1, L_1) \cong \mathcal{O}_F \underset{\mathbb{Z}}{\otimes} R_1$. $\square$

Definition 2.2. *The line bundle* $\omega_1 := \wedge^{\mathrm{top}} \underline{\mathrm{Hom}}_{\mathcal{O}}(L_1, \underline{\mathrm{Lie}}^{\vee}_{A^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}})$ *is called the* **basic automorphic line bundle***.*

A quick fact is:

Lemma 2.3. *We have a canonical isomorphism* $\omega \cong \omega_1^{\otimes t}$ *of invertible sheaves over* $\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}$*.*

A complete theory for automorphic line bundles or vector bundles would require more background (and is beyond this article). We will review these notions and put them in the context of a more comprehensive theory in our forthcoming papers [10] and [11].

We shall focus on the following special cases of automorphic line bundles.

Definition 2.4. *Let* $\Omega^d := \Omega^d_{\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}/\mathsf{S}_1} := \wedge^d \, \Omega^1_{\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}/\mathsf{S}_1}$ *denote the (relative) canonical bundle, and let* $\overline{\Omega}^d := \Omega^d_{\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}/\mathsf{S}_1}(\log \mathsf{D}) \cong \wedge^d \, (\Omega^1_{\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}/\mathsf{S}_1}[\mathrm{d} \log \mathsf{D}])$ *denote the (relative) canonical bundle with logarithmic poles. Note that* $\Omega^d = \overline{\Omega}^d(-\mathsf{D})$*.*

Let $k \geq 0$ *be an integer. The* **automorphic line bundle of cohomological parallel weight** k *is defined to be the line bundle* $\overline{\Omega}^d \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}}}{\otimes} \omega_1^{\otimes k}$*. We shall denote this* **symbolically** *as* $\omega_1^{\otimes (k_c+k)}$*, as if* $\overline{\Omega}^d = \omega_1^{\otimes k_c}$ *for some integer* k_c*. Then we also say that* $\omega_1^{\otimes (k_c+k)}$ *is the automorphic line bundle of parallel weight* $k_c + k$*.*

Remark 2.5. The meaning of $\overline{\Omega}^d$ as an automorphic line bundle can be explained if we introduce vector-valued weights, making k_c meaningful as a vector. It is also possible to identify $\overline{\Omega}^d$ with a positive rational power of ω_1 in $\mathrm{Pic}(\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, making k_c meaningful as a positive rational number, at the expense of losing information on weights (in cases beyond the Siegel and Hilbert–Blumenthal varieties).

For any R_1-algebra, we denote the pullbacks of objects from R_1 to R by attaching the subscript R.

Definition 2.6. *Let* $k \geq 0$ *be an integer, and* R *any* R_1*-algebra. An* R*-valued* **automorphic form** *of cohomological parallel weight* k*, or equivalently of parallel weight* $k_c + k$*, and of level* $\mathcal{H}$*, is an element of the* R*-module*

$$M_{k_c+k}(\mathcal{H}; R) := \Gamma(\mathsf{M}_{\mathcal{H},1,R}^{\mathrm{tor}}, \omega_{1,R}^{\otimes (k_c+k)}) = \Gamma(\mathsf{M}_{\mathcal{H},1,R}^{\mathrm{tor}}, (\overline{\Omega}^d \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}^{\mathrm{tor}}}}{\otimes} \omega_1^{\otimes k})_R).$$

An R-valued automorphic form of parallel weight k_c+k is called **cuspidal**, *in which case we say it is a* **cusp form**, *if it is an element of the R-submodule*

$$S_{k_c+k}(\mathcal{H};R) := \Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1,R}, (\omega_{1,R}^{\otimes(k_c+k)}(-\mathsf{D})) = \Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1,R}, (\Omega^d \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}}{\otimes} \omega_1^{\otimes k})_R)$$

of $M_{k_c+k}(\mathcal{H};R)$ (in other words, it is a global section in $M_{k_c+k}(\mathcal{H};R)$ vanishing along the boundary).

Definition 2.7. *We say that a cohomological parallel weight $k \geq 0$ is* **positive** *if $k > 0$. Equivalently, we say that the parallel weight $k_c + k$ is* **above the least cohomological parallel weight**.

3. Vanishing Theorem of Esnault–Viehweg

Recall the following vanishing theorem in [5]:

Theorem 3.1 (Esnault–Viehweg [5, Prop. 11.5])**.** *Let Z be a proper smooth variety over a perfect field κ of characteristic $p > 0$, E a (possibly non-reduced) effective normal crossings divisor such that E_{red} is a simple normal crossings divisor, and $\mathscr{L}$ an invertible sheaf on Z. Assume the following conditions:*

1. *The triple $(Z, E, \mathscr{L})$ lifts to $(\tilde{Z}, \tilde{E}, \tilde{\mathscr{L}})$ over $W_2(\kappa)$.*
2. *There exists an integer $\nu_0 > 0$ such that, for every integer $\nu \geq \nu_0$, the sheaf $\mathscr{L}^{\otimes \nu}(-E)$ is ample.*

Then, for $a + b < \dim(Z) \leq \mathrm{char}(\kappa)$, one has

$$H^b(Z, \Omega^a_Z(\log E_{\mathrm{red}}) \underset{\mathscr{O}_Z}{\otimes} \mathscr{L}^{\otimes(-1)}) = 0. \tag{3.1}$$

4. Main Results

Recall that the fraction field K_1 of R_1 is a field of characteristic zero and that the residue field κ_1 of R_1 is a field of characteristic p. The mod p cusp forms we will investigate are more precisely the κ_1-valued cusp forms of weight $k_c + k$ for some $k > 0$.

4.1. Vanishing Theorem for Automorphic Line Bundles.

Theorem 4.1. *Suppose that $p = \mathrm{char}(\kappa_1) \geq d$ is a good prime (as in Section 2.1), and let $k > 0$. Then, for all $i > 0$, $H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1,\kappa_1}, (\Omega^d \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}}{\otimes} \omega_1^{\otimes k})_{\kappa_1}) = 0$.*

Proof. By Serre duality, it suffices to show that

$$H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1,\kappa_1}, \omega_{1,\kappa_1}^{\otimes(-k)}) = 0 \tag{4.1}$$

for all $i < d$. For this purpose, we may replace κ_1 with its algebraic closure and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1,\kappa_1}$ with one of its connected components. We apply Theorem 3.1 to $\kappa := \kappa_1$, $Z := \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1,\kappa_1}$, $E := \mathsf{D}'_{\kappa_1}$ (so that $E_{\mathrm{red}} = \mathsf{D}_{\kappa_1}$), and $\mathscr{L} := \omega_{1,\kappa_1}^{\otimes k}$, the dual of the sheaf in (4.1). Since these are pullbacks of objects over R_1 (whose p-adic completion is $W(\kappa_1)$), the first liftability condition (1) is trivially verified. By (2.1) and Lemma 2.3, $\mathscr{L}^{\otimes \nu}(-E)$ is ample for $\nu \geq \nu_0 = r_0 t > 0$. This verifies the second condition (2). Hence (4.1) follows from Theorem 3.1, with $a = 0$ and $b = i < d$. ∎

Remark 4.2. When the Shimura variety in question is either compact or a curve, ω is ample on $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}$, and Theorem 4.1 can be proved (just) with the Kodaira vanishing of Deligne, Illusie, and Raynaud [2, Cor. 2.8], instead of Theorem 3.1.

In the compact case, essentially the same method can be applied to prove that the higher plurigenera (P_m for $m \geq 2$) are invariant in reduction modulo p (cf. [16, Thm 1.2.1(ii)]).

4.2. **Torsion-freeness and Liftability.**

Corollary 4.3. *Suppose $p = \mathrm{char}(\kappa_1) \geq d$ is a good prime, and $k > 0$. Then the following are true:*

(1) *We have an equality*

$$\dim_{K_1}(S_{k_c+k}(\mathcal{H}; K_1)) = \dim_{\kappa_1}(S_{k_c+k}(\mathcal{H}; \kappa_1)). \tag{4.2}$$

(2) *The R_1-module $S_{k_c+k}(\mathcal{H}, R_1)$ is free.*

(3) *Any element of $S_{k_c+k}(\mathcal{H}, \kappa_1)$ is the reduction modulo p of some element in $S_{k_c+k}(\mathcal{H}, R_1)$.*

Proof. All of these are consequences of the proper flatness of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1} \to \mathsf{S}_1$: By upper semi-continuity (cf. [12, §5, Cor. (a)]), Theorem 4.1 shows that, for all $i > 0$, $H^i(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1,K_1}, (\Omega^d \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}}{\otimes} \omega_1^{\otimes k})_{K_1}) = 0$. Hence, (4.2) follows from the invariance of the Euler characteristics (cf. [12, §5, Cor. (b)]). The second and third statements now follow from [12, §5, Cor. 2]. □

Remark 4.4. If we consider cusp forms defined by sections of $\omega_1^{\otimes k'}(-\mathsf{D})$ (instead of $\Omega^d \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}}}{\otimes} \omega_1^{\otimes k}$) and if we regard k_c as a rational number (see Remark 2.5), then the proofs of Theorem 4.1 and Corollary 4.3 carry over as long as $k' > k_c$ (instead of $k > 0$).

Remark 4.5. The bound $k > 0$ in Corollary 4.3 is sharp for Picard modular surfaces. See [16, Thm. 3.4] for counterexamples with $k = 0$.

Remark 4.6. In the special case of modular curves, a lot is already known: For weights ≥ 2 (i.e., $k \geq 0$ in our context), we know all *modular* forms (and not just cusp forms) lift in good reduction, while for weight 1 (which is non-cohomological, or $k = -1$ if appropriately interpreted), we know that some forms do *not* lift. See [3, Lem. 1.9] and [4, Appendix A]. However, the proof of this liftability depends heavily on dimension being 1 (e.g., there is only one higher cohomology and its vanishing follows just from the Riemann–Roch theorem, which takes a particularly simple form for curves) and does not generalize to higher dimensions.

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