COMPACTIFICATIONS OF SPLITTING MODELS OF PEL-TYPE SHIMURA VARIETIES

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Abstract. We construct toroidal and minimal compactifications, with expected properties concerning stratifications and formal local structures, for all integral models of PEL-type Shimura varieties defined by taking normalizations over the splitting models considered by Pappas and Rapoport. (These include, in particular, all the normal flat splitting models they considered.)

Contents

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1. Introduction

In the article [13], we constructed normal flat integral models for all PEL-type Shimura varieties and their toroidal and minimal compactifications constructed by taking normalizations over certain auxiliary choices of good reduction models, with no assumption on the level, ramification, and residue characteristics involved, and showed that such integral models still enjoy many features of the good reduction theory studied as in [5] and [12]. In the article [15], we extended the construction of toroidal compactifications in [13] to allow general projective cone decompositions which are not necessarily induced by the auxiliary choices. When the *local model* M^{loc} for the PEL-type Shimura variety in question is known to be flat over $\mathrm{Spec}(\mathbb{Z}_{(p)})$ and normal, the integral model constructed in [13] coincide with the $\mathscr{A}^{\mathrm{loc}}_{C^p}$ as in [17, (15.4)], which can be interpreted as being constructed by taking normalizations over certain naive models. Thus, the constructions in [13] and [15] provide good toroidal and minimal compactifications for all such integral models.

One naturally also considers the moduli problem $\mathscr{A}^{\mathrm{spl}}_{C^p}$ in the same diagram [17, (15.4)], which corresponds to the *splitting model* $M^{\mathrm{spl}} = \mathscr{M}$ introduced in earlier sections of [17], which are built over $\mathscr{A}^{\mathrm{loc}}_{C^p}$ (over some more naive models) as the relative moduli of certain filtrations on the first de Rham homology of multichains of abelian schemes. For simplicity, let us also call such moduli problems the *splitting models* of the PEL-type Shimura variety. Although they are defined over base rings that are often more ramified, their local properties are often nicer—they do not admit singularities due to restrictions of scalars from ramified extensions. Already in the Hilbert modular case—where the constructions are simple-minded because the splitting models and naive models coincide over the *Rapoport loci* (see [18] and [4]), which are all that are needed for the gluing of boundary charts—the compactifications for splitting models are known to have useful arithmetic applications (see, for example, [22] and [21]).

Our goal is to give a uniform construction, based on [12], [11], [13], and [15], of toroidal and minimal compactifications of all integral models of PEL-type Shimura varieties defined by taking normalizations over such splitting models. These include, in particular, all the normal flat splitting models considered in [17]. But we shall also allow the levels at p to be arbitrarily higher than the stabilizers of the multichains of p-adic lattices used in the definitions of the splitting models.

For the construction of toroidal compactifications of splitting models, the idea is to realize them as *splitting models of toroidal compactifications*. We consider certain filtrations on the *canonical extensions* (over toroidal compactifications of naive models) of the first de Rham homology of multichains of abelian schemes, extending the ones over splitting models. We can show that, over the boundary strata, the normalizations of the relative moduli of such filtrations depend only on the abelian parts of the semi-abelian degenerations, and that their formal boundary charts can be directly built over the formal toroidal boundary charts of the naive models. This allows us to prove a long list of nice properties of such normalizations, including precise descriptions of their stratifications and formal local structures, which allows us to call them *toroidal compactifications* of splitting models.

For the construction of minimal compactifications of splitting models, the conventional approach would be to introduce some variants of the Hodge invertible sheaves, and to consider the projective spectra of the graded algebra formed by sections of their powers. However, there is some subtlety in the choices of such variants. For the projective spectra to define compactifications of our splitting models and admit canonical morphisms from the toroidal compactifications, we need the variants to be *ample* over the splitting models and (at least) *semiample* over the toroidal compactifications; yet we have no a priori knowledge of such variants, except in very special cases. Rather, we will obtain the existence of them as a byproduct of our argument, which is based on a tricky analysis over the formal boundary charts. We will also obtain a long list of nice properties of the corresponding projective spectra, with precise descriptions of their stratifications and of their relation with toroidal compactifications, which allows us to call them *minimal compactifications* of splitting models.

Here is an outline of this article.

Section 2 is devoted to the construction of splitting models of our PEL-type Shimura varieties. In Section 2.1, we review the linear algebraic data for defining multichains of lattices, which are required for the remainder of the article. In Section 2.2, we review the notion of multichains of isogenies of abelian schemes with additional structures; we also introduce their moduli, and relate them to the integral models of PEL-type Shimura varieties constructed by taking normalizations (over certain naive moduli) as in [13]. In Section 2.3, we define the notion of splitting structures, and introduce the relative moduli problems for them. In Section 2.4, we study the splitting structures over the naive moduli and over the integral models of PEL-type Shimura defined by taking normalizations, and introduce their *splitting models*.

Section 3 is devoted to the construction of toroidal compactifications of the splitting models constructed in Section 2. In Section 3.1, we introduce the splitting models over the toroidal compactifications constructed by taking normalizations as in [13] and by normalizations of blowups as in [15], and define the boundary stratification on them. We will consider these the toroidal compactifications of the splitting models. In Sections 3.2 and 3.3, we introduce splitting models over simpler objects over integral models of smaller PEL-type moduli problems associated with the boundary strata, and use them to describe the formal completions of the toroidal compactifications of splitting models along their boundary strata. Theorem 3.3.1 can be considered the technical heart of this article. In Section 3.4, we summarize our main results for toroidal compactifications in Theorem 3.4.1, in a format similar to the one of [12, Thm. 6.4.1.1]. The theorem is rather long, but has the advantage of collecting all relevant information at a single place. We also record some byproducts concerning local properties along the boundary.

Section 4 is devoted to the construction of minimal compactifications of the splitting models constructed in Section 2. In Sections 4.1 and 4.2, we construct them as certain birational contractions of the toroidal compactifications constructed in Section 3, overcoming the difficulty mentioned above. In Section 4.3, we summarize our main results for minimal compactifications in Theorem 4.3.1, in a format similar to the one of [12, Thm. 7.2.4.1].

We shall follow [12, Notation and Conventions] unless otherwise specified. While for practical reasons we cannot explain everything we need from [12], we recommend the reader to make use of the reasonably detailed index and table of contents there, when looking for the numerous definitions. It is not necessary to have completely mastered the techniques in [12], [13], and [15] before reading this article. (Readers who are willing to work with less precise collections of cone decompositions induced by certain auxiliary ones, as in [13, Sec. 7], can ignore most references to [15].)

2. Splitting models

2.1. **Multichains of p-adic lattices.** Suppose we have an integral PEL datum $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$, where $\mathcal{O}$ is an order in a semisimple algebra finite-dimensional over $\mathbb{Q}$, together with a positive involution $\star$, and where $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ is a PEL-type $\mathcal{O}$-lattice as in [12, Def. 1.2.1.3], which defines a group functor G over $\operatorname{Spec}(\mathbb{Z})$ as in [12, Def. 1.2.1.6]. Let us denote the center of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ by F, and denote by F^+ the subalgebra of F consisting of elements invariant under $\star$. Suppose that L satisfies [12, Cond. 1.4.3.10]. (This is harmless in practice, as explained in [12, Rem. 1.4.3.9].)

Let F_0 denote the *reflex field* defined by $(\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ as in [12, Def. 1.2.5.4], which is a subfield of $\mathbb{C}$. Let V_0 (resp. V_0^c) denote the maximal sub-$\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-module of $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ on which $h_0(z)$ acts as $1 \otimes z$ (resp. $1 \otimes z^c$), where c denotes the complex conjugation. Then V_0 and V_0^c are maximal totally isotropic with respect to the pairing $\langle\,\cdot\,,\,\cdot\,\rangle \underset{\mathbb{Z}}{\otimes} \mathbb{C}$, and we have the *Hodge decomposition* $L \underset{\mathbb{Z}}{\otimes} \mathbb{C} \cong V_0 \oplus V_0^c \cong V_0 \oplus V_0^\vee$.

By [12, Def. 1.4.1.4] (with $\Box = \emptyset$ there), for each open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}})$, we have a moduli problem $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{S}_0 = \operatorname{Spec}(F_0)$, defined as the category fibered in groupoids over $(\mathrm{Sch}\,/\mathsf{S}_0)$ whose fiber over each scheme S is the groupoid $\mathsf{M}_{\mathcal{H}}(S)$ described as follows: The objects of $\mathsf{M}_{\mathcal{H}}(S)$ are tuples $(A, \lambda, i, \alpha_{\mathcal{H}})$, where:

(1) $A \to S$ is an abelian scheme.
(2) $\lambda : A \to A^\vee$ is a polarization.
(3) $i : \mathcal{O} \to \operatorname{End}_S(A)$ is an $\mathcal{O}$-endomorphism structure for (A, λ) as in [12, Def. 1.3.3.1].
(4) $\underline{\mathrm{Lie}}_{A/S}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$-module structure given naturally by i satisfies the determinantal condition in [12, Def. 1.3.4.1] given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$.
(5) $\alpha_{\mathcal{H}}$ is an (integral) level-$\mathcal{H}$ structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \langle\,\cdot\,,\,\cdot\,\rangle)$ as in [12, Def. 1.3.7.6].

The morphisms of $\mathsf{M}_{\mathcal{H}}(S)$ are the naive ones induced by isomorphisms between abelian schemes, respecting all the additional structures.

Let $p > 0$ be a rational prime number. For simplicity, and for consistency with [17, Sec. 15], we shall make the following:

Assumption 2.1.1. *The order $\mathcal{O}$ is maximal at p (see* [12, Def. 1.1.1.11]*).*

Let v denote a place of F_0 above p, and let $F_{0,v}$ denote the v-adic completion of F_0. Let $\bar{\mathbb{Q}}$ denote the algebraic closure of F_0 in $\mathbb{C}$, and let $\bar{\mathbb{Q}}_p$ denote an algebraic closure of $F_{0,v}$, with a lifting $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ of the canonical morphism $F_0 \to F_{0,v}$. Let Υ denote the set of homomorphisms $\tau : F \to \bar{\mathbb{Q}}_p$. For each $\tau \in \Upsilon$, let F_τ (resp. F_τ^+) denote the composite of $\mathbb{Q}_p$ and $\tau(F)$ (resp. $\tau(F^+)$) in $\bar{\mathbb{Q}}_p$. We define two $\tau : F \to F_\tau$ and $\tau' : F \to F_{\tau'}$ to be equivalent, denoted $\tau \sim \tau'$, if there exists an isomorphism $\sigma : F_\tau^+ \xrightarrow{\sim} F_{\tau'}^+$ over $\mathbb{Q}_p$ such that $\tau'|_{F^+} = \sigma \circ (\tau|_{F^+})$. In other words, they are equivalent if their restrictions to F^+ are in the same $\operatorname{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-orbit. For each equivalence class $[\tau] \in \Upsilon/\sim$, let us fix the choice of some representative

τ of $[\tau]$, and abusively write $[\tau] : F \to F_{[\tau]}$ and $[\tau] : F^+ \to F^+_{[\tau]}$, where $F^+_{[\tau]} := F^+_\tau$ and $F_{[\tau]} := F \underset{F^+}{\otimes} F^+_\tau$. Then we have a factorization

$$F \underset{\mathbb{Q}}{\otimes} \mathbb{Q}_p \cong \prod_{[\tau]\in\Upsilon/\sim} F_{[\tau]}, \tag{2.1.2}$$

which induces and is induced by a factorization

$$F^+ \underset{\mathbb{Q}}{\otimes} \mathbb{Q}_p \cong \prod_{[\tau]\in\Upsilon/\sim} F^+_{[\tau]} \tag{2.1.3}$$

(cf. [12, Sec. 1.1.2]). These factorizations induce the corresponding factorizations of rings of integers. Since $\mathcal{O}$ is maximal at p by Assumption 2.1.1, it contains the ring $\mathcal{O}_F$ (resp. $\mathcal{O}_{F^+}$) of integers in F (resp. F^+). (We shall always denote by $\mathcal{O}_?$ the ring of integers in any ? that is a product of local or global fields.) Consequently, the identity elements of the rings $\mathcal{O}_{F_{[\tau]}}$ define idempotent elements of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$, and we have a factorization

$$\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p \cong \prod_{[\tau]\in\Upsilon/\sim} \mathcal{O}_{[\tau]}, \tag{2.1.4}$$

inducing for each $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$-module M a canonical decomposition

$$M \cong \underset{[\tau]\in\Upsilon/\sim}{\oplus} M_{[\tau]}, \tag{2.1.5}$$

where each $M_{[\tau]}$ is the maximal submodule of M on which the action of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ (resp. $\mathcal{O}_F$) factors through $\mathcal{O}_{[\tau]}$ (resp. $\mathcal{O}_{F_{[\tau]}}$). In particular, we have a canonical decomposition

$$L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p \cong \underset{[\tau]\in\Upsilon/\sim}{\oplus} L_{[\tau]}, \tag{2.1.6}$$

Let $\mathscr{L}$ be a set of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$-lattices in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ that is a product of sets $\mathscr{L}_{[\tau]}$ of $\mathcal{O}_{[\tau]}$-lattices in $L_{[\tau]} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ in the sense that, for each $\Lambda \in \mathscr{L}$, there exist $\Lambda_{[\tau]} \in \mathscr{L}_{[\tau]}$, for all $[\tau] \in \Upsilon/\sim$, such that

$$\Lambda = \underset{[\tau]\in\Upsilon/\sim}{\oplus} \Lambda_{[\tau]} \tag{2.1.7}$$

as subsets of

$$L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p \cong \underset{[\tau]\in\Upsilon/\sim}{\oplus} (L_{[\tau]} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}). \tag{2.1.8}$$

For simplicity, we shall assume that $\Lambda_0 = L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p \in \mathscr{L}$.

We shall assume moreover that each $\mathscr{L}_{[\tau]}$ is a *chain* in that it satisfies the following two conditions, as in [19, Def. 3.1]:

1. If $\Lambda_{[\tau]}$ and $\Lambda'_{[\tau]}$ are two distinct elements in $\mathscr{L}_{[\tau]}$, then either $\Lambda_{[\tau]} \subsetneq \Lambda'_{[\tau]}$ or $\Lambda'_{[\tau]} \subsetneq \Lambda_{[\tau]}$.
2. If b is a unit of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ which normalizes $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$, then $b\Lambda_{[\tau]} \in \mathscr{L}_{[\tau]}$ for each $\Lambda_{[\tau]} \in \mathscr{L}_{[\tau]}$.

Then $\mathscr{L}$ is a *multichain* as in [19, Def. 3.4]. We shall assume that $\mathscr{L}$ is *self-dual* in the sense that, for each $\Lambda \in \mathscr{L}$, the dual lattice

$$\Lambda^{\#} := \{x \in L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p : \langle x, y\rangle \in \mathbb{Z}_p(1),\ \forall y \in \Lambda\} \tag{2.1.9}$$

is also contained in $\mathscr{L}$ (see [19, Def. 3.13]). As in [19], we shall consider $\mathscr{L}$ as a category with morphisms given by inclusions of lattices.

Definition 2.1.10. *$\mathcal{U}_p(\mathscr{L})$ is the subgroup of $\mathrm{G}(\mathbb{Q}_p)$ consisting of elements stabilizing all lattices Λ in $\mathscr{L}$.*

Remark 2.1.11. By the explanation in [19, 3.2], under the assumption that $\Lambda_0 = L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p \in \mathscr{L}$, we have $\mathcal{U}_p(p) := \ker(\mathrm{G}(\mathbb{Z}_p) \to \mathrm{G}(\mathbb{F}_p)) \subset \mathcal{U}_p(\mathscr{L}) \subset \mathrm{G}(\mathbb{Z}_p)$. In particular, $\mathcal{U}_p(\mathscr{L})$ is an open compact subgroup of $\mathrm{G}(\mathbb{Z}_p)$. (The assumption that $\Lambda_0 = L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p \in \mathscr{L}$ is only made for the sake of simplicity. It is practically harmless for our purpose, thanks to [12, Cor. 1.4.3.8].)

Definition 2.1.12. *Suppose S is a scheme over $\mathrm{Spec}(\mathcal{O}_{F_{0,v}})$. An $\mathscr{L}$***-set of polarized** $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$**-modules** *is a triple $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$, where:*

1. *$\underline{\mathscr{H}} : \Lambda \mapsto \mathscr{H}_\Lambda$ and $\underline{\mathscr{F}} : \Lambda \mapsto \mathscr{F}_\Lambda$ are functors from the category $\mathscr{L}$ (with morphisms being inclusions of lattices) to the category of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$-modules.*
2. *$\underline{j} : \underline{\mathscr{F}} \to \underline{\mathscr{H}}$ is an injective morphism, whose value at each Λ is denoted by $j_\Lambda : \mathscr{F}_\Lambda \to \mathscr{H}_\Lambda$ (which is a morphism of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$-modules).*
3. *For each $\Lambda \in \mathscr{L}$, let us identify $\mathscr{F}_\Lambda$ with an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$-submodule of $\mathscr{H}_\Lambda$, which is its image under the injective morphism j_Λ. Then we require that both $\mathscr{F}_\Lambda$ and $\mathscr{H}_\Lambda/\mathscr{F}_\Lambda$ are finite locally free $\mathscr{O}_S$-modules, and that $\mathscr{H}_\Lambda/\mathscr{F}_\Lambda$ satisfies the determinantal condition in* [12, Def. 1.3.4.1] *given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$.*
4. *For each $\Lambda \in \mathscr{L}$ and each unit b of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ which normalizes $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$, there are* **periodicity isomorphisms** *$\theta^b_{\mathscr{H}_\Lambda} : \mathscr{H}^b_\Lambda \xrightarrow{\sim} \mathscr{H}_{b\Lambda}$ and $\theta^b_{\mathscr{F}_\Lambda} : \mathscr{F}^b_\Lambda \xrightarrow{\sim} \mathscr{F}_{b\Lambda}$ of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$-modules satisfying $j_{b\Lambda} \circ \theta^b_{\mathscr{F}_\Lambda} = \theta^b_{\mathscr{H}_\Lambda} \circ j_\Lambda$, where the superscript b on any $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$-module means conjugating the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$-structure by b^{-1} (i.e., each element $a \in \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ acts by $b^{-1}ab$).*
5. *For each $\Lambda \in \mathscr{L}$, there exists a perfect pairing*

$$(\,\cdot\,,\,\cdot\,)_\Lambda : \mathscr{H}_\Lambda \times \mathscr{H}_{\Lambda^{\#}} \to \mathscr{O}_S(1), \tag{2.1.13}$$

inducing an isomorphism

$$(\,\cdot\,,\,\cdot\,)^*_\Lambda : \mathscr{H}_\Lambda \xrightarrow{\sim} \mathscr{H}^\vee_{\Lambda^{\#}}(1). \tag{2.1.14}$$

Moreover, for each inclusion $\Lambda \subset \Lambda'$ in $\mathscr{L}$, we have the natural compatibility

$$\big(\underline{\mathscr{H}}((\Lambda')^{\#} \to \Lambda^{\#})\big)^\vee \circ (\,\cdot\,,\,\cdot\,)^*_\Lambda = (\,\cdot\,,\,\cdot\,)^*_{\Lambda'} \circ \underline{\mathscr{H}}(\Lambda \to \Lambda').$$

6. *For each $\Lambda \in \mathscr{L}$, the orthogonal complement $\mathscr{F}^\perp_\Lambda$ of $\mathscr{F}_\Lambda$ with respect to the pairing $(\,\cdot\,,\,\cdot\,)_\Lambda$ in* (2.1.13) *coincides with $\mathscr{F}_{\Lambda^{\#}}$ as submodules of $\mathscr{H}_{\Lambda^{\#}}$. Therefore, the isomorphism* (2.1.14) *canonically induces an isomorphism*

$$\mathscr{F}_\Lambda \xrightarrow{\sim} (\mathscr{H}_{\Lambda^{\#}}/\mathscr{F}_{\Lambda^{\#}})^\vee(1). \tag{2.1.15}$$

By definition, we have the following:

Lemma 2.1.16. *Suppose S is a scheme over $\mathrm{Spec}(\mathcal{O}_{F_{0,v}})$, and suppose $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ is an $\mathscr{L}$-set of polarized $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$-modules as in Definition 2.1.12. Then the pullback of $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ to any scheme T over S is an $\mathscr{L}$-set of polarized $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_T$-modules.*

For any $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ as in Definition 2.1.12, we have compatible canonical decompositions

$$\mathscr{H}_\Lambda \cong \bigoplus_{[\tau]\in\Upsilon/\sim} \mathscr{H}_{\Lambda,[\tau]} \tag{2.1.17}$$

and

$$\mathscr{F}_\Lambda \cong \bigoplus_{[\tau]\in\Upsilon/\sim} \mathscr{F}_{\Lambda,[\tau]} \tag{2.1.18}$$

of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$-modules, as in (2.1.5), which induces a collection

$$\{(\underline{\mathscr{H}}_{[\tau]} : \Lambda \mapsto \mathscr{H}_{\Lambda,[\tau]}, \underline{\mathscr{F}}_{[\tau]} : \Lambda \mapsto \mathscr{F}_{\Lambda,[\tau]})\}_{[\tau]\in\Upsilon/\sim} \tag{2.1.19}$$

of functors from the category $\mathscr{L}$ to the category of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$-modules.

2.2. Multichains of isogenies.

For each scheme S, let $\mathrm{AV}^{(p)}_{\mathcal{O}}(S)$ denote the category of abelian schemes A over S equipped with homomorphisms $i : \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)} \to \mathrm{End}_S(A) \underset{\mathbb{Z}}{\otimes} (\mathbb{Z}_{(p)})_S$, whose morphisms are generated by the homomorphisms and all $\mathbb{Z}^\times_{(p)}$-isogenies (see [12, Def. 1.3.1.17] and [19, 6.3]) that are compatible with the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}$-structures. As usual, for each abelian scheme A in $\mathrm{AV}^{(p)}_{\mathcal{O}}(S)$, we consider the dual abelian scheme $A^\vee$ as an object of $\mathrm{AV}^{(p)}_{\mathcal{O}}(S)$, equipped with the homomorphism $i^\vee : \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)} \to \mathrm{End}_S(A^\vee) \underset{\mathbb{Z}}{\otimes} (\mathbb{Z}_{(p)})_S$ defined by $b \mapsto i(b^\star)^\vee$.

Definition 2.2.1. *Given any multichain $\mathscr{L}$ as in Section 2.1, an $\mathscr{L}$-**set of abelian schemes** $\underline{A}$ over S is a functor $\underline{A} : \mathscr{L} \to \mathrm{AV}^{(p)}_{\mathcal{O}}(S) : \Lambda \mapsto A_\Lambda$, equipped with a $\mathbb{Q}^\times$-isogeny $f_{\Lambda,\Lambda'} : A_\Lambda \to A_{\Lambda'}$ for each inclusion $\Lambda \subset \Lambda'$, which is a $(\mathbb{Z}^\times_{(p)})_S$-multiple of an isogeny, compatible with the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}$-structures, satisfying the following two conditions (see [19, Def. 6.5]):*

1. *For each inclusion $\Lambda \subset \Lambda'$ in $\mathscr{L}$, consider $\ker(f_{\Lambda,\Lambda'}[p^\infty]) \subset A_\Lambda[p^\infty]$ (where $f_{\Lambda,\Lambda'}[p^\infty] : A_\Lambda[p^\infty] \to A_{\Lambda'}[p^\infty]$ is defined because $f_{\Lambda,\Lambda'} : A_\Lambda \to A_{\Lambda'}$ is a $(\mathbb{Z}^\times_{(p)})_S$-multiple of an isogeny), which admits an action of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ induced by the action of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}$ on A_Λ, and factorizes as a fiber product*

   $$\ker(f_{\Lambda,\Lambda'}[p^\infty]) \cong \prod_{[\tau]\in\Upsilon/\sim} (\ker(f_{\Lambda,\Lambda'}[p^\infty]))_{[\tau]}$$

   of finite locally free group schemes over S. On the other hand, the inclusion $\Lambda \subset \Lambda'$ induces an inclusion $\Lambda_{[\tau]} \subset \Lambda'_{[\tau]}$. Then the condition is that

   $$\mathrm{rk}_{\mathscr{O}_S}((\ker(f_{\Lambda,\Lambda'}[p^\infty]))_{[\tau]}) = [\Lambda'_{[\tau]} : \Lambda_{[\tau]}].$$

(2) *For each* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(p)}$*-structure* i_Λ *on* A_Λ*, and for each unit* b *of* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ *which normalizes* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(p)}$*, we define a twisted structure* i_Λ^b *by* $i_\Lambda^b(a) = i_\Lambda(b^{-1}ab)$ *for all* $a \in \mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(p)}$*, and we denote abusively* A_Λ^b *for* A_Λ *with such a twisted* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(p)}$*-structure, so that* $i_\Lambda(b)$ *induces a* $\mathbb{Q}^\times$*-isogeny* $[b] : A_\Lambda^b \to A_\Lambda$ *in* $\mathrm{AV}_{\mathcal{O}}^{(p)}(S)$*. Then the condition is that, for each* $b \in (\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Q})^\times \cap (\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(p)})$ *that normalizes* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(p)}$*, there are periodicity isomorphisms* $\theta^b_{A_\Lambda} : A_\Lambda^b \xrightarrow{\sim} A_{b\Lambda}$ *such that* $[b] = f_{\Lambda,b\Lambda} \circ \theta^b_{A_\Lambda}$*.*

Lemma 2.2.2. *For any such* $\underline{A}$*, to define a* $\mathbb{Q}$*-homogeneous principal polarization* $\underline{\lambda}$ *as in* [19, Def. 6.6 and 6.7]*, it suffices to give the following* (*less canonical*) *data:*

(1) *A lattice* $\Lambda_0 \in \mathscr{L}$ *such that* Λ_0 *is contained in its dual lattice* $\Lambda_0^\#$ (*with respect to* $\langle\,\cdot\,,\,\cdot\,\rangle\underset{\mathbb{Z}}{\otimes}\mathbb{Q}_p$). (*Such a* $\Lambda_0 \in \mathscr{L}$ *always exists, by scaling any* $\Lambda \in \mathscr{L}$ *by a sufficiently large power of* p.) *We may and we shall just take* Λ_0 *to be the same* $\Lambda_0 = L\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_p$ *introduced above.*

(2) *A polarization* $\lambda_{\Lambda_0} : A_{\Lambda_0} \to A_{\Lambda_0}^\vee$ *respecting the* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(p)}$*-structures of* A_{Λ_0} *and* $A_{\Lambda_0}^\vee$ *such that, for each* $\Lambda \subset \Lambda_0$*, so that* $\Lambda_0 \subset \Lambda_0^\# \subset \Lambda^\#$*, we have*

$$\ker(f_{\Lambda_0,\Lambda^\#}[p^\infty]) = \ker((f^\vee_{\Lambda,\Lambda_0} \circ \lambda_{\Lambda_0})[p^\infty])$$

in $A_{\Lambda_0}[p^\infty]$ (*where* $f_{\Lambda_0,\Lambda^\#}[p^\infty]$ *and* $(f^\vee_{\Lambda,\Lambda_0} \circ \lambda_{\Lambda_0})[p^\infty]$ *are defined because* $f_{\Lambda_0,\Lambda^\#}$ *and* $f^\vee_{\Lambda,\Lambda_0}$ *are* $(\mathbb{Z}^\times_{(p)})_S$*-multiples of isogenies*)*, so that*

$$f^\vee_{\Lambda,\Lambda_0} \circ \lambda_{\Lambda_0} \circ f^{-1}_{\Lambda_0,\Lambda^\#} : A_{\Lambda^\#} \to A^\vee_\Lambda \tag{2.2.3}$$

is a $\mathbb{Z}^\times_{(p)}$*-isogeny* (*i.e., an isomorphism in the category* $\mathrm{AV}_{\mathcal{O}}^{(p)}(S)$).

Remark 2.2.4. The notation system in Lemma 2.2.2 slightly differs from that in [19, Def. 6.6 and 6.7]—we reserve the symbol λ for the polarizations, rather than for the induced $\mathbb{Z}^\times_{(p)}$-isogenies such as $A_\Lambda \to A^\vee_{\Lambda^\#}$.

Definition 2.2.5. *Let* $\mathcal{H}^p$ *be an open compact subgroup of* $\mathrm{G}(\mathbb{A}^{\infty,p})$*. The moduli problem* $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}$ *over* $\mathrm{Spec}(\mathcal{O}_{F_{0,v}})$ *is defined as the category fibered in groupoids over* $(\mathrm{Sch}\,/\,\mathrm{Spec}(\mathcal{O}_{F_{0,v}}))$ *whose fiber over each scheme* S *is the groupoid* $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}(S)$ *described as follows: The objects of* $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}(S)$ *are tuples* $(\underline{A}, \underline{\lambda}, \underline{i}, \underline{\alpha}_{\mathcal{H}^p})$*, where:*

(1) $\underline{A}$ *is an* $\mathscr{L}$*-set of abelian schemes over* S *as in Definition* 2.2.1*.*
(2) $\underline{\lambda}$ *is a* $\mathbb{Q}$*-homogeneous principal polarization as in* [19, Def. 6.6 and 6.7]*, which can be less canonically defined as in Lemma* 2.2.2*.*
(3) $\underline{i} = \{i_\Lambda\}_{\Lambda\in\mathscr{L}}$ *is a collection of* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$*-structures such that each* i_Λ *gives the* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(p)}$*-structure on* A_Λ (*as an object of* $\mathrm{AV}_{\mathcal{O}}^{(p)}(S)$)*, so that* i_Λ *satisfies the Rosati condition defined by the* $\mathbb{Q}^\times$*-polarization* $f^\vee_{p^r\Lambda,\Lambda_0} \circ \lambda_{\Lambda_0} \circ f_{p^r\Lambda,\Lambda_0}$ (*cf.* [12, Def. 1.3.3.1]) *whenever* $p^r\Lambda \subset \Lambda_0$ *in* $\mathscr{L}$ *for some* $r \in \mathbb{Z}$*.*
(4) *For each* $\Lambda \in \mathscr{L}$*,* $\underline{\mathrm{Lie}}_{A_\Lambda/S}$ *with its* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(p)}$*-module structure given by* i_Λ *satisfies the determinantal condition as in* [12, Def. 1.3.4.1] *given by* $(L\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$*.*

(5) $\underline{\alpha}_{\mathcal{H}^p}$ *is a rational level-*$\mathcal{H}^p$ *structure for* $(\underline{A}, \underline{\lambda}, \underline{i})$, *which can be defined by a rational level-*$\mathcal{H}$ *structure* $[\hat{\alpha}_{\Lambda_0}]_{\mathcal{H}^p}$ *for* $(A_{\Lambda_0}, \lambda_{\Lambda_0}, i_{\Lambda_0})$ *as in* [12, Def. 1.3.8.7] (*with* $\Box = \{p\}$ *there, ignoring the requirement of self-duality of pairings at* p). (*Since the* $\mathbb{Q}^\times$*-isogenies* $f_{\Lambda,\Lambda'} : A_\Lambda \to A_{\Lambda'}$ *induces canonical isomorphisms* $\mathrm{V}^p\, A_{\Lambda,\bar{s}} \xrightarrow{\sim} \mathrm{V}^p\, A_{\Lambda',\bar{s}}$ *of* $\pi_1(S,\bar{s})$*-modules at every geometric point* $\bar{s}$, *we might as well define* $\underline{\alpha}_{\mathcal{H}^p}$ *as a collection* $\{[\hat{\alpha}_\Lambda]_{\mathcal{H}^p}\}_{\Lambda\in\mathscr{L}}$ *whose members are all canonically identified with each other.*)

The morphisms of $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}(S)$ *are the naive ones induced by isomorphisms in the category* $\mathrm{AV}^{(p)}_S$ (*which are induced by* $\mathbb{Z}^\times_{(p)}$*-isogenies between abelian schemes*).

Remark 2.2.6. The moduli problem $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}$ is the same as the ones in [19, Ch. 6] and [17, Sec. 15], although the formulations are slightly different. It generalizes the moduli problem $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}^p}$ in [12, Def. 1.4.2.1], or rather the one in [10, Sec. 5] (which was in the good reduction case, without the consideration of multichains of isogenies).

Lemma 2.2.7. *Let* S *be any scheme over* $\mathrm{Spec}(\mathcal{O}_K)$, *and let* $(\underline{A}, \underline{\lambda}, \underline{i}, \underline{\alpha}_{\mathcal{H}^p})$ *be an object of* $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}(S)$. *Consider the assignments*

$$\underline{\mathscr{H}} : \Lambda \mapsto \mathscr{H}_\Lambda := \underline{H}^{\mathrm{dR}}_1(A_\Lambda/S)$$

and

$$\underline{\mathscr{F}} : \Lambda \mapsto \mathscr{F}_\Lambda := \underline{\mathrm{Lie}}^\vee_{A^\vee_\Lambda/S},$$

and the morphism $\underline{j} : \underline{\mathscr{F}} \to \underline{\mathscr{H}}$ *whose value at each* $\Lambda \in \mathscr{L}$ *is the canonical embedding* $j_\Lambda : \underline{\mathrm{Lie}}^\vee_{A^\vee_\Lambda/S} \to \underline{H}^{\mathrm{dR}}_1(A_\Lambda/S)$ *dual to the last morphism in the canonical short exact sequence* $0 \to \underline{\mathrm{Lie}}^\vee_{A_\Lambda/S} \to \underline{H}^1_{\mathrm{dR}}(A_\Lambda/S) \to \underline{\mathrm{Lie}}_{A^\vee_\Lambda/S} \to 0$ (*see* [2, Lem. 2.5.3]). *Then* $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ *is an* $\mathscr{L}$*-set of polarized* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$*-modules as in Definition* 2.1.12. (*The level structure* $\underline{\alpha}_{\mathcal{H}^p}$ *is not used in the construction of* $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$.)

Proof. For each $\Lambda \in \mathscr{L}$, the desired perfect pairing as in (2.1.13) is induced by the canonical perfect pairing $\underline{H}^{\mathrm{dR}}_1(A_\Lambda/S) \times \underline{H}^{\mathrm{dR}}_1(A^\vee_\Lambda/S) \to \mathscr{O}_S(1)$ (see [4, 1.5]), and by the canonical isomorphism $\underline{H}^{\mathrm{dR}}_1(A_{\Lambda^\#}/S) \xrightarrow{\sim} \underline{H}^{\mathrm{dR}}_1(A^\vee_\Lambda/S)$ induced by $\underline{\lambda}$ (or, concretely, by (2.2.3)). The other conditions in Definition 2.1.12 then follow from the various conditions in Definitions 2.2.1 and 2.2.5. ☐

Remark 2.2.8. Since $\underline{H}^{\mathrm{dR}}_1(A_\Lambda/S)$ is canonically isomorphic to the relative Lie algebra of the universal vectorial extension of A_Λ over S (see [16, Ch. 1, Sec. 4]), the $\mathscr{H}_\Lambda$ and $\mathscr{F}_\Lambda$ in Lemma 2.2.7 are the M_Λ and F_Λ in [17, Sec. 15], respectively.

Choices 2.2.9. By the explanation in [19, 3.2], there exists a finite subset $\mathscr{L}_{\mathrm{J}} = \{\Lambda_{\mathrm{j}}\}_{\mathrm{j}\in\mathrm{J}}$ of $\mathscr{L}$ such that an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$-lattice Λ in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ belongs to $\mathscr{L}$ if and only if there exist some integers $(r_{[\tau]})_{[\tau]\in\Upsilon/\sim}$ and $\mathrm{j} \in \mathrm{J}$ such that $\Lambda_{[\tau]} = p^{r_{[\tau]}}\Lambda_{\mathrm{j},[\tau]}$, for all $[\tau] \in \Upsilon/\sim$, where $\Lambda_{[\tau]}$ and $\Lambda_{\mathrm{j},[\tau]}$ are the direct factors of Λ and Λ_{j}, respectively, as in (2.1.7). Take any $r_0 \in \mathbb{Z}$ such that $\Lambda_{\mathrm{j}} \subset p^{r_0}\Lambda_0$ for all $\mathrm{j} \in \mathrm{J}$. Then there exists a set $\{L_{\mathrm{j}}\}_{\mathrm{j}\in\mathrm{J}}$ of $\mathcal{O}$-lattices in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ such that $L_{\mathrm{j}} \subset p^{r_0}L$, such that the canonical morphism $(p^{r_0}L)/L_{\mathrm{j}} \to (p^{r_0}L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p)/(L_{\mathrm{j}} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p)$ is an isomorphism of $\mathcal{O}$-modules, and such that $L_{\mathrm{j}} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p = \Lambda_{\mathrm{j}}$ in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$, for all $\mathrm{j} \in \mathrm{J}$. Let $g_{\mathrm{j}} = 1$, and let $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{j}}$ be the restriction of $p^{-2r_0}\langle\,\cdot\,,\,\cdot\,\rangle$ to L_{j}, for each $\mathrm{j} \in \mathrm{J}$. For each $\mathrm{j} \in \mathrm{J}$, since $\mathcal{O}$ is maximal at p by Assumption 2.1.1, and since $L_{\mathrm{j}} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p = L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p$, the lattice L_{j}

satisfies [12, Cond. 1.4.3.10] as L does. Moreover, if $\mathcal{H}^p$ is any subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^p)$, whose action stabilizes $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p$ by definition, then it also stabilizes $L_{\mathrm{j}} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p$. From now on, we shall fix the choices of J and $\mathscr{L}_{\mathrm{J}} = \{\Lambda_{\mathrm{j}}\}_{\mathrm{j} \in \mathrm{J}}$.

Choices 2.2.10. Let us take $\mathcal{H}$ to be any open compact subgroup of $\mathrm{G}(\mathbb{A}^\infty)$ such that its image $\mathcal{H}^p$ under the canonical homomorphism $\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\hat{\mathbb{Z}}^p)$ is a *neat* (see [12, Def. 1.4.1.8]) open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^p)$, in which case $\mathcal{H}$ is also neat, and such that the image $\mathcal{H}_p$ of $\mathcal{H}$ under the canonical homomorphism $\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\mathbb{Z}_p)$ is contained in $\mathcal{U}_p(\mathscr{L})$ as in Definition 2.1.10 (see also Remark 2.1.11). Then the collection $\{(1, L_{\mathrm{j}}, \langle \cdot\,, \cdot \rangle_{\mathrm{j}})\}_{\mathrm{j} \in \mathrm{J}}$ satisfies the requirements in [13, Sec. 2], and we can define $\vec{\mathsf{M}}_{\mathcal{H}}$ as in [13, Prop. 6.1] (by taking normalization over a product of minimal compactifications of auxiliary good reduction integral models indexed by j).

Proposition 2.2.11. *Let $\mathcal{H}$ and $\mathcal{H}^p$ be as in Choices 2.2.10. Then there is a canonical finite étale morphism*

$$\mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} F_{0,v} \to \mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \tag{2.2.12}$$

over $\mathrm{Spec}(F_{0,v})$*, which is an open and closed immersion when $\mathcal{H}$ is of the form $\mathcal{H}^p\mathcal{U}_p(\mathscr{L})$, which extends to a canonical finite morphism*

$$\vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_{F_{0,v}} \to \mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p} \tag{2.2.13}$$

over $\mathrm{Spec}(\mathcal{O}_{F_{0,v}})$.

Proof. Since the canonical morphism $\mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}^p\mathcal{U}_p(\mathscr{L})}$ is finite étale, and since the induced canonical morphism $\vec{\mathsf{M}}_{\mathcal{H}} \to \vec{\mathsf{M}}_{\mathcal{H}^p\mathcal{U}_p(\mathscr{L})}$ is finite (essentially by definition), we may and we shall assume that $\mathcal{H} = \mathcal{H}^p\mathcal{U}_p(\mathscr{L})$ in the remainder of the proof.

Consider the pullback to $S := \mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} F_{0,v}$ of the tautological tuple over $\mathsf{M}_{\mathcal{H}}$, which we abusively denote by $(A, \lambda, i, \alpha_{\mathcal{H}})$. For each $\mathrm{j} \in \mathrm{J}$, we also have the pullback to S of the tautological tuple over $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$, via the canonical isomorphism $\mathsf{M}_{\mathcal{H}} \overset{\sim}{\to} \mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$ given by [13, (2.1)], which we abusively denote by $(A_{\mathrm{j}}, \lambda_{\mathrm{j}}, i_{\mathrm{j}}, \alpha_{\mathcal{H}_{\mathrm{j}}})$. By [13, Prop. 6.1], for each $\mathrm{j} \in \mathrm{J}$, the triple $(A_{\mathrm{j}}, \lambda_{\mathrm{j}}, i_{\mathrm{j}})$ over S extends to a triple $(\vec{A}_{\mathrm{j}}, \vec{\lambda}_{\mathrm{j}}, \vec{i}_{\mathrm{j}})$ over $\vec{S} := \vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_{F_{0,v}}$. By [7, IV-2, 6.8.2 and 7.8.3], $\vec{S}$ is noetherian normal, because $\vec{\mathsf{M}}_{\mathcal{H}}$ is of finite type over $\mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ and normal. By the proof of [13, (2.1)] based on [12, Prop. 1.4.3.4 and Cor. 1.4.3.8], for any two $\mathrm{j}, \mathrm{j}' \in \mathrm{J}$, there canonically exists a $\mathbb{Q}^\times$-isogeny $f_{\mathrm{j},\mathrm{j}'} : A_{\mathrm{j}} \to A_{\mathrm{j}'}$ over S. By [12, Prop. 3.3.1.5] and the noetherian normality of $\vec{S}$, it (uniquely) extends to a $\mathbb{Q}^\times$-isogeny $\vec{f}_{\mathrm{j},\mathrm{j}'} : \vec{A}_{\mathrm{j}} \to \vec{A}_{\mathrm{j}'}$ over $\vec{S}$. Hence, for any $\mathrm{j} \in \mathrm{J}$, with $\Lambda_{\mathrm{j}} = L_{\mathrm{j}} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ in $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$, and for any $r \in \mathbb{Z}$, we can define $A_{p^r\Lambda_{\mathrm{j}}}$ to be the abelian scheme $\vec{A}_{\mathrm{j}}$ over $\vec{S}$. In general, for each $\Lambda \in \mathscr{L}$ such that $\Lambda_{[\tau]} = p^{r_{[\tau]}}\Lambda_{\mathrm{j},[\tau]}$ for some integers $(r_{[\tau]})_{[\tau] \in \Upsilon/\sim}$ and $\mathrm{j} \in \mathrm{J}$, for all $[\tau] \in \Upsilon/\sim$, as in Choices 2.2.9, there exists some $r \in \mathbb{Z}$ such that $r \geq r_{[\tau]}$, for all $[\tau] \in \Upsilon/\sim$, in which case we have a finite locally free subgroup scheme $\mathcal{K} := \prod_{[\tau] \in \Upsilon/\sim} (\vec{A}_{\mathrm{j}}[p^{r - r_{[\tau]}}])_{[\tau]}$ of $\vec{A}_{\mathrm{j}}$ over $\vec{S}$, and we can define A_Λ to be the abelian scheme $\vec{A}_{\mathrm{j}}/\mathcal{K}$ over $\vec{S}$, with a canonically induced isogeny $f_{p^r\Lambda_{\mathrm{j}},\Lambda} : A_{p^r\Lambda_{\mathrm{j}}} \to A_\Lambda$. For any $\Lambda' \in \mathscr{L}$ such that $\Lambda \subset \Lambda'$ and $\Lambda'_{[\tau]} = p^{r'_{[\tau]}}\Lambda_{\mathrm{j}',[\tau]}$ for some integers $(r'_{[\tau]})_{[\tau] \in \Upsilon/\sim}$

and j$'$ $\in$ J, for all $[\tau] \in \Upsilon/\sim$, as in Choices 2.2.9, so that we have a similarly defined isogeny $f_{p^{r'}\Lambda_{\mathrm{j}'},\Lambda'} : A_{p^{r'}\Lambda_{\mathrm{j}'}} \to A_{\Lambda'}$, we define $f_{\Lambda,\Lambda'} : A_\Lambda \to A_{\Lambda'}$ to be the $\mathbb{Q}^\times$-isogeny given by the composition of $f_{p^{r'}\Lambda_{\mathrm{j}'},\Lambda'} \circ \vec{f}_{\mathrm{j},\mathrm{j}'} \circ f^{-1}_{p^r\Lambda_{\mathrm{j}},\Lambda}$ with multiplication by $p^{r-r'}$ on $A_{\Lambda'}$. At any geometric point $\bar{s} \to S$, the level structures $\alpha_{\mathcal{H}_{\mathrm{j}}}$ and $\alpha_{\mathcal{H}_{\mathrm{j}'}}$ compatibly induce isomorphisms matching the submodules $(L_{\mathrm{j}} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p) \times \Lambda$ and $(L_{\mathrm{j}'} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p) \times \Lambda'$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty \cong (L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,p}) \times (L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p)$ with the submodules $\mathrm{T}\, A_{\Lambda,\bar{s}}$ and $\mathrm{T}\, A_{\Lambda',\bar{s}}$ of $\mathrm{V}\, A_{\bar{s}}$, respectively, so that the conditions in Definition 2.2.1 holds over the open dense subscheme S of $\vec{S}$, and therefore also over the whole $\vec{S}$. Thus, the assignments above define an $\mathscr{L}$-set $\underline{A}$ of abelian schemes over $\vec{S}$, as in Definition 2.2.1.

For any $\mathrm{j}_0 \in \mathrm{J}$, since $\Lambda_{\mathrm{j}_0} \subset p^{r_0}\Lambda_0$ (see Choices 2.2.9), we have an isogeny $f_{p^{-r_0}\Lambda_{\mathrm{j}_0},\Lambda_0} : A_{p^{-r_0}\Lambda_{\mathrm{j}_0}} = \vec{A}_{\mathrm{j}_0} \to A_{\Lambda_0}$, as in the previous paragraph, and we can define the $\mathbb{Q}^\times$-polarization $\lambda_{\Lambda_0} : A_{\Lambda_0} \to A^\vee_{\Lambda_0}$ to be $(f^\vee_{p^{-r_0}\Lambda_{\mathrm{j}_0},\Lambda_0})^{-1} \circ \vec{\lambda}_{\mathrm{j}_0} \circ f^{-1}_{p^{-r_0}\Lambda_{\mathrm{j}_0},\Lambda_0}$. Since the level structure $\alpha_{\mathcal{H}_{\mathrm{j}_0}}$ matches the submodules $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p) \times \Lambda_0$ and $(L^\# \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p) \times \Lambda_0^\#$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\infty \cong (L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,p}) \times (L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p)$ with the submodules $\mathrm{T}\, A_{\Lambda_0,\bar{s}}$ and $\mathrm{T}\, A^\vee_{\Lambda_0,\bar{s}}$ of $\mathrm{V}\, A_{\bar{s}}$, respectively, for each geometric point $\bar{s} \to S$, and since $\Lambda_0 \subset \Lambda_0^\#$ (see Lemma 2.2.2), the $\mathbb{Q}^\times$-isogeny λ_{Λ_0} defined above is a $\mathbb{Z}^\times_{(p)}$-multiple of an isogeny over S, and hence is also a $\mathbb{Z}^\times_{(p)}$-multiple of an isogeny over $\vec{S}$, again by [12, Prop. 3.3.1.5] and the noetherian normality of $\vec{S}$. By Lemma 2.2.2, we have also obtained a $\mathbb{Q}$-homogeneous principal polarization $\underline{\lambda}$ for $\underline{A}$ as in [19, Def. 6.6 and 6.7].

The $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}$-structure $\underline{i} = \{i_\Lambda\}_{\Lambda\in\mathscr{L}}$ for $(\underline{A}, \underline{\lambda})$ is compatibly induced by the $\mathcal{O}$-endomorphism structures $\vec{i}_{\mathrm{j}}$ for $(\vec{A}_{\mathrm{j}}, \vec{\lambda}_{\mathrm{j}})$, for all $\mathrm{j} \in \mathrm{J}$.

For each $\Lambda \in \mathscr{L}$, the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\vec{S}}$-module $\underline{\mathrm{Lie}}_{A_\Lambda/\vec{S}}$ satisfies the determinantal condition as in [12, Def. 1.3.4.1] defined by the data $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ because the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$-modules $\underline{\mathrm{Lie}}_{A_{\mathrm{j}}/S}$ do, for all $\mathrm{j} \in \mathrm{J}$, over the open dense subscheme S of $\vec{S}$, and because the determinantal condition is a closed condition by definition.

Finally, by forgetting the factors at p, the level structures $\alpha_{\mathcal{H}_{\mathrm{j}}}$ over S compatibly induce the level structures $\alpha_{\mathcal{H}^p_{\mathrm{j}}}$ away from p, realized by compatible collections of subschemes $\alpha_{\mathrm{j},n}$ of $\underline{\mathrm{Hom}}_S((L_{\mathrm{j}}/nL_{\mathrm{j}})_S, A_{\mathrm{j}}[n]) \underset{S}{\times} \underline{\mathrm{Hom}}_S(((\mathbb{Z}/n\mathbb{Z})(1))_S, \boldsymbol{\mu}_{n,S})$ finite étale over S, which are étale-locally-defined orbits of symplectic isomorphisms, for sufficiently divisible integers n prime to p. Since $\vec{S}$ is noetherian and normal, they uniquely extend to compatible collections of subschemes $\vec{\alpha}_{\mathrm{j},n}$ of $\underline{\mathrm{Hom}}_{\vec{S}}((L_{\mathrm{j}}/nL_{\mathrm{j}})_{\vec{S}}, \vec{A}_{\mathrm{j}}[n]) \underset{\vec{S}}{\times} \underline{\mathrm{Hom}}_{\vec{S}}(((\mathbb{Z}/n\mathbb{Z})(1))_{\vec{S}}, \boldsymbol{\mu}_{n,\vec{S}})$ finite étale over $\vec{S}$, which define level structures $\vec{\alpha}_{\mathcal{H}^p_{\mathrm{j}}}$ away from p and induce the desired level-$\mathcal{H}^p$ structure $\underline{\alpha}_{\mathcal{H}^p}$ for $(\underline{A}, \underline{\lambda}, \underline{i})$ (by the same argument as in [12, Constr. 1.3.8.4 and Rem. 1.3.8.9]).

Thus we have obtained a tuple $(\underline{G}, \underline{\lambda}, \underline{i}, \underline{\alpha}_{\mathcal{H}^p})$ over $\vec{S}$ which is parameterized by $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}$, which induces a morphism $\vec{S} \to \mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}$ as in (2.2.13). Since the construction of the canonical finite morphism $\vec{S} \to \prod_{\mathrm{j}\in\mathrm{J}} \mathsf{M}_{\mathcal{H}_{\mathrm{j},\mathrm{aux}}}$ given by [13, (6.3)] only uses level structures away from p, by rewriting the objects of $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}$ represented by $\mathbb{Z}^\times_{(p)}$-isogeny classes in terms of isomorphism classes by the same argument as in the proof of [12, Prop. 1.4.3.3], it factors as a composition $\vec{S} \to \mathsf{M}_{\mathcal{H}^p}^{\mathrm{naive}} \to \prod_{\mathrm{j}\in\mathrm{J}} \mathsf{M}_{\mathcal{H}_{\mathrm{j,aux}}}$, where the first morphism $\vec{S} \to \mathsf{M}_{\mathcal{H}^p}^{\mathrm{naive}}$ is (2.2.13). This shows that (2.2.13) is also finite.

By restriction to S, we obtain a finite morphism $S \to \mathsf{M}_{\mathcal{H}^p}^{\mathrm{naive}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ as in (2.2.12). By comparing their universal properties, both sides of (2.2.12) admit compatible morphisms to $S^p := \mathsf{M}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} F_{0,v}$. By assumption, $\mathcal{U}_p(\mathscr{L})$ (see Definition 2.1.10) is the subgroup of $\mathrm{G}(\mathbb{Q}_p)$ consisting of elements stabilizing all lattices Λ in $\mathscr{L}$, which is also the subgroup of $\mathrm{G}(\mathbb{Q}_p)$ consisting of elements stabilizing all the submodules L_{j} of $p^{r_0}L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$, for all $\mathrm{j} \in \mathrm{J}$ (see Choices 2.2.9). Take any $r_1 \in \mathbb{Z}$ such that $p^{r_1}L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p \subset L_{\mathrm{j}} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p \subset p^{r_0}L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ for all $\mathrm{j} \in \mathrm{J}$. Since the morphism $S \to S^p$ is the pullback of the canonical morphism $\mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} F_0$, which is defined by forgetting the level structures at p, it parameterizes étale-locally-defined orbits of symplectic isomorphisms of the form $((p^{r_0}L)/(p^{r_1}L))_S \xrightarrow{\sim} A[p^{r_1-r_0}]$ over S, under which the images of $(L_{\mathrm{j}}/(p^{r_1}L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p))_S$ determine some isogenies $A \to A_{\mathrm{j}}$, satisfying some additional conditions which are open and closed. On the other hand, the morphism $\mathsf{M}_{\mathcal{H}^p}^{\mathrm{naive}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to S^p$ parameterizes exactly such isogenies $A \to A_{\mathrm{j}}$ satisfying some other closed conditions. Hence, by comparing the relative universal properties, the morphism (2.2.12) is an open and closed immersion (under the simplified assumption that $\mathcal{H} = \mathcal{H}^p\mathcal{U}_p(\mathscr{L})$), as desired. $\square$

2.3. **Splitting structures and their relative moduli.**

Choices 2.3.1. For each equivalence class $[\tau] \in \Upsilon/\sim$, let us order the elements $\tau_{[\tau],0}, \tau_{[\tau],1}, \ldots, \tau_{[\tau],i}, \ldots$ in $[\tau]$, where the index i satisfies $0 \leq i < d_{[\tau]} := [F_{[\tau]} : \mathbb{Q}_p]$, in a way such that any two elements with the same restriction to F^+ are successive. Let K be any finite extension of $\mathbb{Q}_p$ in $\bar{\mathbb{Q}}_p$ that contains the composite of F_τ in $\bar{\mathbb{Q}}_p$ for all $\tau \in \Upsilon$, namely the composite of $\mathbb{Q}_p$ and the Galois closure of F in $\bar{\mathbb{Q}}_p$. Then $F_{0,v} \subset K$ (cf. the proof of [12, Cor. 1.2.5.7]). We shall fix the choices of K and of the orderings $\tau_{[\tau],0}, \tau_{[\tau],1}, \ldots, \tau_{[\tau],i}, \ldots$, from now on.

Let $\{r_\tau\}_{\tau\in\Upsilon}$ be integers such that, for every $b \in F$, we have

$$\det(T - b \cdot \mathrm{Id}_{V_0^\vee} | V_0^\vee) = \prod_{\tau\in\Upsilon} (T - \tau(b))^{r_\tau} \tag{2.3.2}$$

in $\bar{\mathbb{Q}}_p[T]$, as in [17, Sec. 14]. (As explained in [17, Sec. 14], for every $\tau \in \Upsilon$, the $F_{[\tau]}$-module $L_{[\tau]} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is necessarily free of rank $r_\tau + r_{\tau\circ\star}$.)

Definition 2.3.3. *Suppose that S is a scheme over* $\mathrm{Spec}(\mathcal{O}_K)$*, and that* $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ *is an $\mathscr{L}$-set of polarized $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$-modules as in Definition* 2.1.12*, which induces as in* (2.1.19) *the collection* $\{(\underline{\mathscr{H}}_{[\tau]}, \underline{\mathscr{F}}_{[\tau]})\}_{[\tau]\in\Upsilon/\sim}$. *A* **splitting structure** *for* $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ *is a collection*

$$\{(\underline{\mathscr{F}}_{[\tau]}^i, \underline{j}_{[\tau]}^i)\}_{[\tau]\in\Upsilon/\sim, 0\leq i<d_{[\tau]}}, \tag{2.3.4}$$

where each $\underline{\mathscr{F}}_{[\tau]}^i : \Lambda \mapsto \mathscr{F}_{\Lambda,[\tau]}^i$ is a functor from the category $\mathscr{L}$ to the category of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$-modules, and where each $\underline{j}_{[\tau]}^i : \underline{\mathscr{F}}_{[\tau]}^i \to \underline{\mathscr{H}}_{[\tau]}$ is an injective morphism, *whose value at each* Λ *is denoted by* $j^i_{\Lambda,[\tau]} : \mathscr{F}^i_{\Lambda,[\tau]} \to \mathscr{H}_{\Lambda,[\tau]}$*, which satisfies the following conditions:*

(1) *For each* $\Lambda \in \mathscr{L}$*, let us identify* $\mathscr{F}^i_{\Lambda,[\tau]}$ *with an* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$*-submodule of* $\mathscr{H}_{\Lambda,[\tau]}$*, which is its image under the injective morphism* $j^i_{\Lambda,[\tau]}$*. Then we require that both* $\mathscr{F}^i_{\Lambda,[\tau]}$ *and* $\mathscr{H}_{\Lambda,[\tau]}/\mathscr{F}^i_{\Lambda,[\tau]}$ *are finite locally free* $\mathscr{O}_S$*-modules.*

(2) *For each* $\Lambda \in \mathscr{L}$*, we have*

$$0 = \mathscr{F}^{d_{[\tau]}}_{\Lambda,[\tau]} \subset \mathscr{F}^{d_{[\tau]}-1}_{\Lambda,[\tau]} \subset \cdots \subset \mathscr{F}^1_{\Lambda,[\tau]} \subset \mathscr{F}^0_{\Lambda,[\tau]} = \mathscr{F}_{\Lambda,[\tau]}$$

as $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$*-submodule of* $\mathscr{H}_{\Lambda,[\tau]}$*, where* $\mathscr{F}_{\Lambda,[\tau]}$ *is as in* (2.1.18)*. For each integer* i *satisfying* $0 \leq i < d_{[\tau]}$*, the quotient* $\mathscr{F}^i_{\Lambda,[\tau]}/\mathscr{F}^{i+1}_{\Lambda,[\tau]}$ *is a locally free* $\mathscr{O}_S$*-module of rank* $r_{\tau_{[\tau],i}}$ *annihilated by* $b \otimes 1 - 1 \otimes \tau_{[\tau],i}(b)$ *for all* $b \in \mathcal{O}_{F_{[\tau]}}$*.*

(3) *For each* $\Lambda \in \mathscr{L}$*, each* $[\tau] \in \Upsilon/\sim$*, each integer* i *satisfying* $0 \leq i < d_{[\tau]}$*, and each unit* b *of* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p$ *which normalizes* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$*, there are* **periodicity isomorphisms** $\theta^b_{\mathscr{F}^i_{\Lambda,[\tau]}} : (\mathscr{F}^i_{\Lambda,[\tau]})^b \xrightarrow{\sim} \mathscr{F}^i_{b\Lambda,[\tau]}$ *of* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$*-modules satisfying* $j^i_{[\tau],b\Lambda} \circ \theta^b_{\mathscr{F}^i_{\Lambda,[\tau]}} = \theta^b_{\mathscr{H}_{\Lambda,[\tau]}} \circ j^i_{\Lambda,[\tau]}$ *(where the superscript* b *on an* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$*-module means conjugating the* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$*-structure by* b^{-1}*, as in Definition* 2.1.12*).*

(4) *For each* $\Lambda \in \mathscr{L}$ *and each integer* i *satisfying* $0 \leq i < d_{[\tau]}$*, let* $(\mathscr{F}^i_{\Lambda,[\tau]})^\perp$ *denote the orthogonal complement of* $\mathscr{F}^i_{\Lambda,[\tau]}$ *in* $\mathscr{H}_{\Lambda^\#,[\tau]}$ *with respect to the perfect pairing* $\mathscr{H}_{\Lambda,[\tau]} \times \mathscr{H}_{\Lambda^\#,[\tau]} \to \mathscr{O}_S(1)$ *induced by the perfect pairing* (2.1.13)*, which satisfies* $\mathscr{F}^i_{\Lambda^\#,[\tau]} \subset \mathscr{F}_{\Lambda^\#,[\tau]} = \mathscr{F}^\perp_{\Lambda,[\tau]} \subset (\mathscr{F}^i_{\Lambda,[\tau]})^\perp$*. Then*

$$\prod_{0 \leq k < i} (b \otimes 1 - 1 \otimes \tau_{[\tau],k}(b))((\mathscr{F}^i_{\Lambda,[\tau]})^\perp) \subset \mathscr{F}^i_{\Lambda^\#,[\tau]}$$

for all $b \in \mathcal{O}_{F_{[\tau]}}$*, for every* $0 < i \leq d_{[\tau]}$ *divisible by* $[F_{[\tau]} : F^+_{[\tau]}]$*.*

Definition 2.3.5. *Two splitting structures*

$$\{(\underline{\mathscr{F}}^i_{[\tau]}, \underline{j}^i_{[\tau]})\}_{[\tau] \in \Upsilon/\sim, 0 \leq i < d_{[\tau]}}$$

and

$$\{(\underline{\mathscr{F}}^{i,\prime}_{[\tau]}, \underline{j}^{i,\prime}_{[\tau]})\}_{[\tau] \in \Upsilon/\sim, 0 \leq i < d_{[\tau]}}$$

as in Definition 2.3.3 *are* **isomorphic** *to each other if there exist isomorphisms* $\rho^i_{[\tau]} : \underline{\mathscr{F}}^i_{[\tau]} \xrightarrow{\sim} \underline{\mathscr{F}}^{i,\prime}_{[\tau]}$ *such that* $\underline{j}^{i,\prime}_{[\tau]} \circ \rho^i_{[\tau]} = \underline{j}^i_{[\tau]}$ *for all* $[\tau] \in \Upsilon/\sim$ *and* $0 \leq i < d_{[\tau]}$*.*

By definition, we have the following:

Lemma 2.3.6. *Suppose that* S *is a scheme over* $\mathrm{Spec}(\mathcal{O}_K)$*, that* $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ *is an* $\mathscr{L}$*-set of polarized* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$*-modules as in Definition* 2.1.12*, and that*

$$\{(\underline{\mathscr{F}}^i_{[\tau]}, \underline{j}^i_{[\tau]})\}_{[\tau] \in \Upsilon/\sim, 0 \leq i < d_{[\tau]}}$$

is a splitting structure for $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ *as in Definition* 2.3.3*. Then the pullback of* $\{(\underline{\mathscr{F}}^i_{[\tau]}, \underline{j}^i_{[\tau]})\}_{[\tau] \in \Upsilon/\sim, 0 \leq i < d_{[\tau]}}$ *to any scheme* T *over* S *is a splitting structure for the pullback of* $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ *to* T *(cf. Lemma* 2.1.16*).*

Proposition 2.3.7. *Consider the (contravariant) functor*

$$\mathrm{Spl}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S} : (\mathrm{Sch}/S) \to (\mathrm{Sets})$$

defined by assigning to each scheme T over S the set of isomorphism classes of splittings structures for the pullback of $(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})$ to T. Then the functor $\mathrm{Spl}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ is representable by a scheme over S, which we abusively denote by the same symbols. This scheme is locally over S projective, with a relatively ample invertible sheaf given by the relative Hodge invertible sheaf

$$\omega^{\underline{\mu}}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S} := \bigotimes_{\Lambda\in\mathscr{L}_{\mathrm{J}}}\Bigg(\bigotimes_{[\tau]\in\Upsilon/\sim}\Bigg(\bigotimes_{0\leq i<d_{[\tau]}}\big(\wedge^{\mathrm{top}}\,(\mathscr{F}^i_{\Lambda,[\tau]})\big)^{\otimes(\mu^i_{\Lambda,[\tau]}-\mu^{i-1}_{\Lambda,[\tau]})}\Bigg)\Bigg) \tag{2.3.8}$$

(with the convention that $\mu^{-1}_{\Lambda,[\tau]}=0$), where $\mathscr{L}_{\mathrm{J}}$ is the subset of $\mathscr{L}$ as in Choices 2.2.9, and where the tensor and exterior products are over $\mathscr{O}_{\mathrm{Spl}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}}$, for each triply indexed collection of integers $\underline{\mu}=\{\mu^i_{\Lambda,[\tau]}\}_{\Lambda\in\mathscr{L}_{\mathrm{J}},[\tau]\in\Upsilon/\sim,0\leq i<d_{[\tau]}}$ that is **positive** *in the sense that $\mu^{i-1}_{\Lambda,[\tau]}>\mu^i_{\Lambda,[\tau]}$ for all $\Lambda\in\mathscr{L}_{\mathrm{J}}$, $[\tau]\in\Upsilon/\sim$, and $0<i<d_{[\tau]}$.*

Proof. For simplicity, let us abusively denote by the same symbols the pullback of $(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})$ to any scheme T over S. Let $\{(\underline{\mathscr{F}}^i_{[\tau]},\underline{j}^i_{[\tau]})\}_{[\tau]\in\Upsilon/\sim,0\leq i<d_{[\tau]}}$ be a splitting structure for $(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})$. As in Definitions 2.1.12 and 2.3.3, let us identify $\mathscr{F}_{\Lambda,[\tau]}$ with a submodule of $\mathscr{H}_{\Lambda,[\tau]}$ via $j_{\Lambda,[\tau]}$, and identify $\mathscr{F}^i_{\Lambda,[\tau]}$ with a submodule of $\mathscr{F}_{\Lambda,[\tau]}$ via $j^i_{\Lambda,[\tau]}$, for all $\Lambda\in\mathscr{L}$, $[\tau]\in\Upsilon/\sim$, and $0\leq i<d_{[\tau]}$. Then the splitting structure is uniquely determined by the filtrations defined by $\{\mathscr{F}^i_{\Lambda,[\tau]}\}_{0\leq i<d_{[\tau]}}$ on $\mathscr{H}_{\Lambda,[\tau]}$, for all $\Lambda\in\mathscr{L}$ and $[\tau]\in\Upsilon/\sim$, satisfying the additional conditions in Definition 2.3.3. By the periodicity condition (3) in Definition 2.3.3, and by the same explanation as in [19, 3.2], it suffices to consider the indices $\Lambda\in\mathscr{L}_{\mathrm{J}}$, as in Choices 2.2.9.

Locally over the base scheme S, the filtered pieces $\mathscr{F}^i_{\Lambda,[\tau]}$ of $\mathscr{H}_{\Lambda,[\tau]}$, which are $\mathscr{O}_S$-module local direct summands by assumption, are parameterized by some Grassmannians; and the inclusion relations $\mathscr{F}^{i+1}_{\Lambda,[\tau]}\subset\mathscr{F}^i_{\Lambda,[\tau]}$ are given by the vanishing of the canonical morphisms $\mathscr{F}^{i+1}_{\Lambda,[\tau]}\to\mathscr{F}_{\Lambda,[\tau]}/\mathscr{F}^i_{\Lambda,[\tau]}$, which are closed conditions. Similarly, the additional conditions in Definition 2.3.3, given by the containment of images of certain morphisms, are also closed conditions. Hence, $\mathrm{Spl}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ is locally over S representable by a closed subscheme in the fiber product of Grassmannians triply indexed by the finitely many $\Lambda\in\mathscr{L}_{\mathrm{J}}$, $[\tau]\in\Upsilon/\sim$, and $0<i<d_{[\tau]}$. As explained in, for example, [6, Sec. 5.1.6], the Grassmannian triply indexed by Λ, $[\tau]$, and i has an ample invertible sheaf whose pullback to $\mathrm{Spl}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ is tautologically dual to $\wedge^{\mathrm{top}}\,(\mathscr{F}^i_{\Lambda,[\tau]})$, the top exterior power of the locally free sheaf $\mathscr{F}^i_{\Lambda,[\tau]}$ over $\mathscr{O}_{\mathrm{Spl}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}}$. Since each such $\wedge^{\mathrm{top}}\,(\mathscr{F}^i_{\Lambda,[\tau]})$ is globally defined over $\mathrm{Spl}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$, and since $\wedge^{\mathrm{top}}\,(\mathscr{F}^0_{\Lambda,[\tau]})$ descends to S because $\mathscr{F}^0_{\Lambda,[\tau]}=\mathscr{F}_{\Lambda,[\tau]}$ does, the scheme $\mathrm{Spl}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ is locally over S projective, with a relatively ample invertible sheaf $\omega^{\underline{\mu}}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ given by (2.3.8) for each positive $\underline{\mu}$. □

Lemma 2.3.9. *Suppose that S is a scheme over $\mathrm{Spec}(K)$, and that $(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})$ is an $\mathscr{L}$-set of polarized $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathscr{O}_S$-modules as in Definition 2.1.12. Then the structural morphism $\mathrm{Spl}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}\to S$ is an isomorphism. Equivalently, for each scheme T over S, there is up to isomorphism a unique splitting structure for the pullback of $(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})$ to T. Moreover, the condition (4) in Definition 2.3.3 is redundant.*

Proof. Let us proceed as in the proof of Proposition 2.3.7, with the addition assumption that $T = \mathrm{Spec}(R)$ is affine; it suffices to show that there uniquely exist filtrations $\{\mathscr{F}^i_{\Lambda,[\tau]}\}_{0\leq i<d_{[\tau]}}$ on $\mathscr{H}_{\Lambda,[\tau]}$ satisfying the additional conditions in Definition 2.3.3, for all $\Lambda \in \mathscr{L}$ and $[\tau] \in \Upsilon/\sim$. Since $F_{[\tau]} \underset{\mathbb{Q}_p}{\otimes} K \cong \prod_{\tau\in[\tau]} K_\tau = \prod_{0\leq i<d_{[\tau]}} K_{\tau_{[\tau],i}}$, where F acts on $K_\tau = K$ via the homomorphism $\tau : F \to K$, we have canonical decompositions $\mathscr{F}_{\Lambda,[\tau]} \cong \underset{0\leq i<d_{[\tau]}}{\oplus} \mathscr{F}_{\Lambda,\tau_{[\tau],i}}$ and $\mathscr{H}_{\Lambda,[\tau]}/\mathscr{F}_{\Lambda,[\tau]} \cong \mathscr{F}^\vee_{\Lambda^\#,[\tau]} \cong \underset{0\leq i<d_{[\tau]}}{\oplus} \mathscr{F}^\vee_{\Lambda^\#,\tau_{[\tau],i}}$ of $\mathcal{O}_{F_{[\tau]}} \underset{\mathbb{Z}_p}{\otimes} R \cong F_{[\tau]} \underset{\mathbb{Q}_p}{\otimes} R$-modules, which are (up to permutation) independent of the ordering $\tau_{[\tau],0}, \tau_{[\tau],1}, \ldots$ of elements in $[\tau]$. Hence, the desired filtration $\{\mathscr{F}^i_{\Lambda,[\tau]}\}_{0\leq i<d_{[\tau]}}$ on $\mathscr{F}_{\Lambda,[\tau]}$ uniquely exists and is given by $\mathscr{F}^i_{\Lambda,[\tau]} \cong \underset{i\leq k<d_{[\tau]}}{\oplus} \mathscr{F}_{\Lambda,\tau_{[\tau],k}}$, which satisfies $(\mathscr{F}^i_{\Lambda^\#,[\tau]})^\perp/\mathscr{F}_{\Lambda,[\tau]} \cong \underset{0\leq k<i}{\oplus} \mathscr{F}^\vee_{\Lambda^\#,\tau_{[\tau],k}}$ for all $0 \leq i < d_{[\tau]}$. In particular, the condition (4) in Definition 2.3.3 is redundant. □

Proposition 2.3.10. *Consider also the functor* $\mathrm{Spl}'_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ *defined by assigning to each scheme T over S the set of isomorphism classes of splittings structures for the pullback of* $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ *to T, but without the last condition* (4)*. By the proof of Proposition* 2.3.7*,* $\mathrm{Spl}'_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ *is representable by a scheme over S, which is locally over S projective, and the canonical forgetful morphism*

$$\mathrm{Spl}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S} \to \mathrm{Spl}'_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S} \tag{2.3.11}$$

is a closed immersion, under which the invertible sheaf $\omega^{\underline{\mu}}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ *defined over* $\mathrm{Spl}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ *(see* (2.3.8)*) is the pullback of a similarly defined invertible sheaf* $\omega^{\underline{\mu},\prime}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ *over* $\mathrm{Spl}'_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$*, which is also relatively ample over S, for each positive* $\underline{\mu}$*.*

Suppose moreover that $S \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *is* **reduced**. *By Lemma* 2.3.9*, the morphisms*

$$\mathrm{Spl}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to \mathrm{Spl}'_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to S \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \tag{2.3.12}$$

canonically induced by (2.3.11) *are both isomorphisms. Therefore, if we denote by* $\mathrm{Spl}^+_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ *(resp.* $\mathrm{Spl}'^{,+}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$*) the normalization of the* (*necessarily reduced*) *schematic closure of* $S \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *in* $\mathrm{Spl}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ *(resp.* $\mathrm{Spl}'_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$*) via such canonical isomorphisms, then* (2.3.11) *canonically induces an isomorphism*

$$\mathrm{Spl}^+_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S} \xrightarrow{\sim} \mathrm{Spl}'^{,+}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}. \tag{2.3.13}$$

We shall denote the pullback of $\omega^{\underline{\mu}}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ *(or* $\omega^{\underline{\mu},\prime}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$*) to* $\mathrm{Spl}^+_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ *by* $\omega^{\underline{\mu},+}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$*, which is relatively ample over S because the canonical normalization morphism* $\mathrm{Spl}^+_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S} \to \mathrm{Spl}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/S}$ *is finite, for each positive* $\underline{\mu}$*.*

Proof. The statements are self-explanatory. □

2.4. **Splitting models for PEL moduli.**

Let $\mathcal{H}$ and $\mathcal{H}^p$ be as in Choices 2.2.10.

Definition 2.4.1 (cf. [17, the end of Sec. 15])**.** *Let* $\mathcal{H}^p$ *be an open compact subgroup of* $\mathrm{G}(\mathbb{A}^{\infty,p})$*. The moduli problem* $\mathsf{M}^{\mathrm{spl}}_{\mathcal{H}^p}$ *over* $\mathrm{Spec}(\mathcal{O}_K)$ *is defined as the category* *fibered in groupoids over* $(\mathrm{Sch}\,/\,\mathrm{Spec}(\mathcal{O}_K))$ *whose fiber over each scheme* S *is the groupoid* $\mathsf{M}^{\mathrm{spl}}_{\mathcal{H}^p}(S)$ *described as follows: The objects of* $\mathsf{M}^{\mathrm{spl}}_{\mathcal{H}^p}(S)$ *are tuples*

$$(\underline{A}, \underline{\lambda}, \underline{i}, \underline{\alpha}_{\mathcal{H}^p}, \{(\underline{\mathscr{F}}^i_{[\tau]}, \underline{j}^i_{[\tau]})\}_{[\tau]\in\Upsilon/\sim, 0\leq i<d_{[\tau]}}),$$

where $(\underline{A}, \underline{\lambda}, \underline{i}, \underline{\alpha}_{\mathcal{H}^p})$ *is an object of* $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}(S)$ *as in Definition* 2.2.5*, and where* $\{(\underline{\mathscr{F}}^i_{[\tau]}, \underline{j}^i_{[\tau]})\}_{[\tau]\in\Upsilon/\sim, 0\leq i<d_{[\tau]}}$ *is a splitting structure* (*as in Definition* 2.3.3) *for the* $\mathscr{L}$*-set* $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ *of polarized* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$*-modules associated with* $(\underline{A}, \underline{\lambda}, \underline{i})$ *as in Lemma* 2.2.7*. The morphisms of* $\mathsf{M}^{\mathrm{spl}}_{\mathcal{H}^p}(S)$ *are the naive ones induced by isomorphisms in the category* $\mathrm{AV}^{(p)}_{\mathcal{O}}(S)$ (*given by* $\mathbb{Z}^\times_{(p)}$*-isogenies between abelian schemes with* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}$*-structures*) *and by the isomorphisms between splitting structures as in Definition* 2.3.5*.*

Then Proposition 2.3.7 implies the following:

Lemma 2.4.2. *The canonical morphism*

$$\mathsf{M}^{\mathrm{spl}}_{\mathcal{H}^p} \to \mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_{0,v}}}{\otimes} \mathcal{O}_K \tag{2.4.3}$$

defined by forgetting splitting structures is relatively representable and projective. If we abusively denote by $(\underline{A}, \underline{\lambda}, \underline{i})$ *the pullback to* $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_{0,v}}}{\otimes} \mathcal{O}_K$ *of* (*part of*) *the tautological object over* $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}$*, and denote by* $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ *the associated* $\mathscr{L}$*-set of polarized* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$*-modules as in Lemma* 2.2.7*, then we have a canonical isomorphism*

$$\mathsf{M}^{\mathrm{spl}}_{\mathcal{H}^p} \xrightarrow{\sim} \mathrm{Spl}_{(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})/(\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p} \underset{\mathcal{O}_{F_{0,v}}}{\otimes} \mathcal{O}_K)}. \tag{2.4.4}$$

Definition 2.4.5. *Let* $(\underline{A}, \underline{\lambda}, \underline{i})$ *abusively denote the pullback to* $\vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K$ *of the tautological object over* $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}$ *under the morphism* (2.2.13)*, and let* $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ *denote the associated* $\mathscr{L}$*-set of polarized* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$*-modules as in Lemma* 2.2.7*. Then we define* (*as in Proposition* 2.3.10)

$$\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} := \mathrm{Spl}^+_{(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})/(\vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K)}. \tag{2.4.6}$$

Lemma 2.4.7. *The canonical morphism* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to \vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} K \cong \mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} K$ *induced by the structural morphism* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K$ *is an isomorphism.*

Proof. This follows from Lemma 2.3.9. □

Corollary 2.4.8. *Let* $\mathsf{M}^{\mathrm{loc}}_{\mathcal{H}^p}$ *denote the schematic image of the canonical morphism* $\mathsf{M}^{\mathrm{spl}}_{\mathcal{H}^p} \to \mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}$ *induced by* (2.4.3)*. Then the morphism* (2.2.13) *factors through the structural closed immersion* $\mathsf{M}^{\mathrm{loc}}_{\mathcal{H}^p} \hookrightarrow \mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}$ *and induces a canonical finite morphism*

$$\vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_{F_{0,v}} \to \mathsf{M}^{\mathrm{loc}}_{\mathcal{H}^p} \tag{2.4.9}$$

over $\mathrm{Spec}(\mathcal{O}_{F_{0,v}})$*, extending the finite étale morphism* (2.2.12) *over* $\mathrm{Spec}(F_{0,v})$*. If* (2.2.12) *is an open and closed immersion, and if* $\mathsf{M}^{\mathrm{loc}}_{\mathcal{H}^p}$ *is known to be flat over* $\mathrm{Spec}(\mathcal{O}_{F_{0,v}})$ *and normal, then* (2.4.9) *is an open and closed immersion.*

Corollary 2.4.10. *Suppose that the morphism* (2.4.9) *induced by* (2.2.13) *is an open and closed immersion, and that* $\mathsf{M}^{\mathrm{spl}}_{\mathcal{H}^p}$ *is known to be flat over* $\operatorname{Spec}(\mathcal{O}_K)$ *and normal. Then we have a canonical isomorphism*

$$\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \xrightarrow{\sim} (\vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_{F_{0,v}}) \underset{\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}}{\times} \mathsf{M}^{\mathrm{spl}}_{\mathcal{H}^p}, \tag{2.4.11}$$

inducing an open and closed immersion

$$\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{spl}}_{\mathcal{H}^p} \tag{2.4.12}$$

compatible with (2.4.9)*. If* (2.4.9) *is an isomorphism, then so is* (2.4.12)*.*

Remark 2.4.13. To summarize, we have a commutative diagram:

$$\begin{array}{ccccccc} \mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} & \longrightarrow & \mathsf{M}^{\mathrm{spl}}_{\mathcal{H}^p} & & \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \\ \mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} F_{0,v} & \hookrightarrow & \vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_{F_{0,v}} & \longrightarrow & \mathsf{M}^{\mathrm{loc}}_{\mathcal{H}^p} & \hookrightarrow & \mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p} \end{array} \tag{2.4.14}$$

By Proposition 2.3.7 and Lemma 2.4.2, and by their definitions, the vertical morphisms are all projective and surjective—the left-most one is finite étale (and is just the base change morphism). The two horizontal arrows at the left-hand side are open immersions with schematically dense images, by definition. By Corollaries 2.4.8 and 2.4.10, if (2.2.12) is an open and closed immersion (which is the case when $\mathcal{H} = \mathcal{H}^p\mathcal{U}_p(\mathscr{L})$, by Proposition 2.2.11), and if $\mathsf{M}^{\mathrm{loc}}_{\mathcal{H}^p}$ and $\mathsf{M}^{\mathrm{spl}}_{\mathcal{H}^p}$ are known to be flat over $\operatorname{Spec}(\mathcal{O}_{F_{0,v}})$ and $\operatorname{Spec}(\mathcal{O}_K)$, respectively, and are both known to be normal, then the horizontal arrows between the two middle columns are open and closed immersions. By definition, the bottom-right arrow is a closed immersion.

Remark 2.4.15. The $\mathsf{M}^{\mathrm{spl}}_{\mathcal{H}^p}$, $\mathsf{M}^{\mathrm{loc}}_{\mathcal{H}^p}$, and $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}$ in (2.4.14) are what were denoted $\mathscr{A}^{\mathrm{spl}}_{C^p}$, $\mathscr{A}^{\mathrm{loc}}_{C^p}$, and $\mathscr{A}^{\mathrm{naive}}_{C^p}$ in [17, (15.4)], respectively, where the latter three objects have the same singularities as the splitting model $\mathscr{M}$, the local model M^{loc}, and the naive local model M^{naive}, respectively, defined and studied there. While they will play no role in the remaining constructions of this article, they are important for practical applications of the results in this article.

Remark 2.4.16. The normality of $\mathsf{M}^{\mathrm{loc}}_{\mathcal{H}^p}$ and $\mathsf{M}^{\mathrm{spl}}_{\mathcal{H}^p}$, and their flatness over $\operatorname{Spec}(\mathcal{O}_{F_{0,v}})$ and $\operatorname{Spec}(\mathcal{O}_K)$, respectively, are known in many cases. See, for example, [17].

Proposition 2.4.17 (cf. [13, Prop. 13.1 and 13.15])**.** *Suppose that* $\mathcal{H}$ *and* $\mathcal{H}'$ *are two open compact subgroups of* $\mathrm{G}(\hat{\mathbb{Z}})$ *such that their images under the canonical homomorphism* $\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\mathbb{Z}_p)$ *are contained in* $\mathcal{U}_p(\mathscr{L})$ *as in Definition* 2.1.10*; that* $g \in \mathrm{G}(\mathbb{A}^\infty)$ *is an element such that the multiplication by the image* g_p *of* g *under the canonical homomorphism* $\mathrm{G}(\mathbb{A}^\infty) \to \mathrm{G}(\mathbb{Q}_p)$ *preserves the multichain* $\mathscr{L}$*; and that* $\mathcal{H} \subset g\mathcal{H}'g^{-1}$*. Then we have a canonical projective morphism*

$$\vec{[g]} : \vec{\mathsf{M}}_{\mathcal{H}} \to \vec{\mathsf{M}}_{\mathcal{H}'} \tag{2.4.18}$$

extending the canonical finite morphism $\mathsf{M}_{\mathcal{H}} \xrightarrow{\sim} \mathsf{M}_{g^{-1}\mathcal{H}g} \to \mathsf{M}_{\mathcal{H}'}$ *defined by* g*, whose pullback from* $\mathcal{O}_{F_0,(p)}$ *to* $\mathcal{O}_K$ *lifts to a canonical projective morphism*

$$\vec{[g]}^{\mathrm{spl}} : \vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}'}. \tag{2.4.19}$$

Proof. In this proof, as in [13, Sec. 13] and [15, Sec. 7], for the sake of clarity, let us temporarily (and abusively) denote all objects constructed using $\{(1, L_{\mathrm{j}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{j}})\}_{\mathrm{j}\in\mathrm{J}}$ (see Choices 2.2.9 and 2.2.10) by an additional subscript J. Since multiplication by g_p preserves the multichain $\mathscr{L}$, by [13, (2.1)] (or rather by its proof based on [12, Prop. 1.4.3.4 and Cor. 1.4.3.8]), the tautological objects over $\vec{\mathsf{M}}_{\mathcal{H},\{0,1\}\times\mathrm{J}}$ (as in [13, Ex. 13.14]) differ from those over $\vec{\mathsf{M}}_{\mathcal{H},\mathrm{J}}$ by repeating some of the latter by Hecke twists by the image g^p of g under the canonical homomorphism $\mathrm{G}(\mathbb{A}^\infty) \to \mathrm{G}(\mathbb{A}^{\infty,p})$, realized by $\mathbb{Z}_{(p)}^\times$-isogenies, up to shifting the indices. Therefore, we have $\vec{\mathsf{M}}_{\mathcal{H},\mathrm{J}} \cong \vec{\mathsf{M}}_{\mathcal{H},\{0,1\}\times\mathrm{J}} \cong \vec{\mathsf{M}}_{g^{-1}\mathcal{H}g,\mathrm{J}}$, and the composition of these with the canonical morphism $\vec{\mathsf{M}}_{g^{-1}\mathcal{H}g,\mathrm{J}} \to \vec{\mathsf{M}}_{\mathcal{H}',\mathrm{J}}$ gives the desired (2.4.18). Moreover, the pullback under (2.4.18) of the tautological $\mathscr{L}$-set of polarized $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$-modules over $\vec{\mathsf{M}}_{\mathcal{H}',\mathrm{J}}$ can be identified (up to shifting the indices) with the one over $\vec{\mathsf{M}}_{\mathcal{H},\mathrm{J}}$ via an isomorphism canonically induced by g, and so (2.4.18) induces the desired (2.4.19), because the two sides of (2.4.19) are the respective normalizations of relative moduli for splitting structures over the base changes of the two sides of (2.4.18) from $\mathcal{O}_{F_0,(p)}$ to $\mathcal{O}_K$ (and by Zariski's main theorem; see [7, III-1, 4.4.3, 4.4.11]). □

3. Toroidal compactifications

3.1. **Splitting models for toroidal compactifications.** Let $\mathcal{H}$ be as in Choices 2.2.10, and let $\vec{\mathsf{M}}_{\mathcal{H}} \hookrightarrow \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ be any toroidal compactification as in either [13, (7.10)] or [15, Thm. 6.1]. Let $(\underline{A}, \underline{\lambda}, \underline{i})$ abusively denote the pullback to $\vec{\mathsf{M}}_{\mathcal{H}}$ of the tautological object over $\mathsf{M}^{\mathrm{naive}}_{\mathcal{H}^p}$, under the morphism (2.2.13), and let $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ denote the associated $\mathscr{L}$-set of polarized $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\vec{\mathsf{M}}_{\mathcal{H}}}$-modules as in Lemma 2.2.7.

Lemma 3.1.1. *For each $\Lambda \in \mathscr{L}$, the abelian scheme A_Λ (resp. $A^\vee_\Lambda$) over $\vec{\mathsf{M}}_{\mathcal{H}}$ (necessarily uniquely) extends to a semi-abelian scheme A^{ext}_Λ (resp. $A^{\mathrm{ext},\vee}_\Lambda$) over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$* (*cf.* [12, Thm. 3.4.3.2 and Prop. 3.3.1.5]). *Consequently, by* [12, Prop. 3.3.1.5], *for each inclusion $\Lambda \subset \Lambda'$ in $\mathscr{L}$, the $\mathbb{Q}^\times$-isogeny $f_{\Lambda,\Lambda'} : A_\Lambda \to A_{\Lambda'}$ over $\vec{\mathsf{M}}_{\mathcal{H}}$, which is a $\mathbb{Z}_{(p)}^\times$-multiple of an isogeny, (necessarily uniquely) extends to a $\mathbb{Q}^\times$-isogeny $f^{\mathrm{ext}}_{\Lambda,\Lambda'} : A^{\mathrm{ext}}_\Lambda \to A^{\mathrm{ext}}_{\Lambda'}$ over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, which is also a $\mathbb{Z}_{(p)}^\times$-multiple of an isogeny.*

Proof. By [12, Lem. 3.4.3.1 and Prop. 3.3.1.5], any $\mathbb{Z}_{(p)}^\times$-isogeny of abelian schemes over $\mathsf{M}_{\mathcal{H}}$ (uniquely) extends to a $\mathbb{Z}_{(p)}^\times$-isogeny of semi-abelian schemes over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ as soon as the source extends. Hence, the assertion of the lemma does not depend on the choice of A_Λ in its $\mathbb{Z}_{(p)}^\times$-isogeny class. Therefore, as in the proof of Proposition 2.2.11, for each $\Lambda \in \mathscr{L}$ such that $\Lambda_{[\tau]} = p^{r_{[\tau]}}\Lambda_{\mathrm{j},[\tau]}$ for some integers $(r_{[\tau]})_{[\tau]\in\Upsilon/\sim}$ and $\mathrm{j} \in \mathrm{J}$, for all $[\tau] \in \Upsilon/\sim$, as in Choices 2.2.9, and for $r \in \mathbb{Z}$ such that $r \geq r_{[\tau]}$, for all $[\tau] \in \Upsilon/\sim$, we can take A_Λ to be $\vec{A}_{\mathrm{j}}/\mathcal{K}$, where $\mathcal{K} = \prod\limits_{[\tau]\in\Upsilon/\sim} (\vec{A}_{\mathrm{j}}[p^{r-r_{[\tau]}}])_{[\tau]}$. Since $\vec{A}_{\mathrm{j}}$ extends to a semi-abelian scheme $\vec{A}^{\mathrm{ext}}_{\mathrm{j}}$ with additional structures over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ by [13, Thm. 11.2] and [15, Thm. 6.1], $\mathcal{K}$ also extends to the closed subgroup scheme $\mathcal{K}^{\mathrm{ext}} := \prod\limits_{[\tau]\in\Upsilon/\sim} (\vec{A}^{\mathrm{ext}}_{\mathrm{j}}[p^{r-r_{[\tau]}}])_{[\tau]}$ of $\vec{A}^{\mathrm{ext}}_{\mathrm{j}}$, which is quasi-finite and flat over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. Thus, we can define A^{ext}_Λ to be $\vec{A}^{\mathrm{ext}}_{\mathrm{j}}/\mathcal{K}^{\mathrm{ext}}$, by [12, Lem. 3.4.3.1, Prop. 3.3.1.5, and the same local argument as in the proof of Thm. 3.4.3.2]. □

Proposition 3.1.2. *The $\mathscr{L}$-set $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ of polarized $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\vec{\mathsf{M}}_{\mathcal{H}}}$-modules introduced above (necessarily uniquely) extends to an $\mathscr{L}$-set $(\underline{\mathscr{H}}^{\mathrm{ext}}, \underline{\mathscr{F}}^{\mathrm{ext}}, \underline{j}^{\mathrm{ext}})$ of polarized $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$-modules inducing compatible isomorphisms $\mathscr{F}^{\mathrm{ext}}_{\Lambda} \cong \underline{\mathrm{Lie}}^{\vee}_{A^{\mathrm{ext},\vee}_{\Lambda}/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ and $\mathscr{H}^{\mathrm{ext}}_{\Lambda}/\mathscr{F}^{\mathrm{ext}}_{\Lambda} \cong \underline{\mathrm{Lie}}_{A^{\mathrm{ext}}_{\Lambda}/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ (of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$-modules) extending the canonical isomorphisms $\mathscr{F}_{\Lambda} \cong \underline{\mathrm{Lie}}^{\vee}_{A^{\vee}_{\Lambda}/\vec{\mathsf{M}}_{\mathcal{H}}}$ and $\mathscr{H}_{\Lambda}/\mathscr{F}_{\Lambda} \cong \underline{\mathrm{Lie}}_{A_{\Lambda}/\vec{\mathsf{M}}_{\mathcal{H}}}$ (of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\vec{\mathsf{M}}_{\mathcal{H}}}$-modules), respectively, for all $\Lambda \in \mathscr{L}$.*

Proof. In the proof of Lemma 3.1.1, the quotient $\vec{A}^{\mathrm{ext}}_{\mathrm{j}} \to A^{\mathrm{ext}}_{\Lambda} = \vec{A}^{\mathrm{ext}}_{\mathrm{j}}/\mathcal{K}^{\mathrm{ext}}$, where $\mathcal{K}^{\mathrm{ext}} = \prod_{[\tau]\in\Upsilon/\sim} (\vec{A}^{\mathrm{ext}}_{\mathrm{j}}[p^{r-r_{[\tau]}}])_{[\tau]}$, induces morphisms $(\underline{\mathrm{Lie}}^{\vee}_{\vec{A}^{\mathrm{ext},\vee}_{\mathrm{j}}/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}})_{[\tau]} \to \underline{\mathrm{Lie}}^{\vee}_{A^{\mathrm{ext},\vee}_{\Lambda}/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ and $\underline{\mathrm{Lie}}_{\vec{A}^{\mathrm{ext}}_{\mathrm{j}}/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} \to \underline{\mathrm{Lie}}_{A^{\mathrm{ext}}_{\Lambda}/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ that can be canonically identified with multiplication by $p^{r-r_{[\tau]}}$ on $(\underline{\mathrm{Lie}}^{\vee}_{\vec{A}^{\mathrm{ext},\vee}_{\mathrm{j}}/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}})_{[\tau]}$ and $\underline{\mathrm{Lie}}_{\vec{A}^{\mathrm{ext}}_{\mathrm{j}}/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$, respectively, for each $[\tau] \in \Upsilon/\sim$. Thus, by decomposing everything into factors indexed by $[\tau] \in \Upsilon/\sim$ as in Section 2.1, the proposition follows from [15, Prop. 7.15] (which was based on a reduction first to the case where Σ is induced by auxiliary choices as in [13, Sec. 7], and then to the good reduction case as in [11, Prop. 6.9]). □

Definition 3.1.3. *Let $(\underline{\mathscr{H}}^{\mathrm{ext}}, \underline{\mathscr{F}}^{\mathrm{ext}}, \underline{j}^{\mathrm{ext}})$ be as in Proposition 3.1.2. Then we define*

$$\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} := \mathrm{Spl}^{+}_{(\underline{\mathscr{H}}^{\mathrm{ext}}, \underline{\mathscr{F}}^{\mathrm{ext}}, \underline{j}^{\mathrm{ext}})/(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K)}, \tag{3.1.4}$$

where $\mathrm{Spl}^{+}_{(\underline{\mathscr{H}}^{\mathrm{ext}}, \underline{\mathscr{F}}^{\mathrm{ext}}, \underline{j}^{\mathrm{ext}})/(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K)}$ *is defined as in Proposition 2.3.10.*

By comparing the universal properties (see Definitions 2.4.5 and 3.1.3), we have a canonical morphism

$$\mathrm{Spl}_{(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})/(\vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K)} \hookrightarrow \mathrm{Spl}_{(\underline{\mathscr{H}}^{\mathrm{ext}}, \underline{\mathscr{F}}^{\mathrm{ext}}, \underline{j}^{\mathrm{ext}})/(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K)} \tag{3.1.5}$$

over $\mathrm{Spec}(\mathcal{O}_K)$. By Proposition 2.3.10, (3.1.5) induces a canonical morphism

$$\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \hookrightarrow \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \tag{3.1.6}$$

over $\mathrm{Spec}(\mathcal{O}_K)$, which covers the canonical morphism $\vec{\mathsf{M}}_{\mathcal{H}} \hookrightarrow \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (see [13, (7.10)] and [15, Thm. 6.1]).

Remark 3.1.7. We would like to view $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ as the *toroidal compactification* of $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ associated with the compatible collection Σ of cone decompositions. However, to justify this, we need to show that it satisfies some reasonable properties as in [12, Thm. 6.4.1.1] (and in the corresponding theorems in [13] and [15]).

Definition 3.1.8. *For each (locally closed) stratum $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ as in* [13, Thm. 9.13] *and* [15, Thm. 6.1(3)], *we denote by $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ the reduced subscheme of the preimage of $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ under the canonical morphism $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$.*

Then $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ is a disjoint union of locally closed subschemes

$$\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}, \tag{3.1.9}$$

as in [13, Thm. 9.13] and [15, Thm. 6.1(3)], except that we still have to show that it is a stratification. (As in [12, Thm. 6.4.1.1(2)], the notation "$\coprod$" only means a set-theoretic disjoint union. The algebro-geometric structure is still that of $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$.)

Our next goal will be to understand $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ and the formal completion $(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ of $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ along $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$, for each $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$. (As in [12, Thm. 6.4.1.1(5)], to form the formal completion along a given locally closed subscheme, we first remove the complement of it in its closure in the total space, and then form the formal completion of the remaining space along this stratum.)

3.2. **Toroidal boundary charts and formal completions.** Suppose we have a representative $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ of $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ as in [12, Def. 6.2.6.1], where the underlying $(\mathsf{Z}_{\mathcal{H}},\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ is a representative of cusp label for $\mathsf{M}_{\mathcal{H}}$ as in [12, Def. 5.4.2.4] (where $\mathsf{Z}_{\mathcal{H}}$ is often suppressed in the notation, by [12, Conv. 5.4.2.5]), and where $\sigma\in\Sigma_{\Phi_{\mathcal{H}}}\in\Sigma$ is a cone such that $\sigma\subset\mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$. Consider the schemes $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$, $\vec{\mathsf{M}}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}$, $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$, and $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$, and the formal scheme $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$, defined as in [13, Prop. 7.4 and Sec. 8] and [15, Constr. 4.5].

Definition 3.2.1. *As in Definition* 2.4.5*, let us set*

$$\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}:=\mathrm{Spl}^{+}_{(\,^{\sharp}\underline{\mathscr{H}},\,^{\sharp}\underline{\mathscr{F}},\,^{\sharp}\underline{j})/(\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}\underset{\mathcal{O}_{F_0,(p)}}{\otimes}\mathcal{O}_K)},\tag{3.2.2}$$

where we denote by $(\,^{\sharp}\underline{\mathscr{H}},\,^{\sharp}\underline{\mathscr{F}},\,^{\sharp}\underline{j})$ *the analogue of* $(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})$ *associated with the tautological tuple* $(\underline{B},\underline{\lambda_B},\underline{i_B})$ *over* $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$*, as in Lemma* 2.2.7*, and abusively denote by the same symbols its pullbacks to schemes and formal schemes over* $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$*, such as* $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}\underset{\mathcal{O}_{F_0,(p)}}{\otimes}\mathcal{O}_K$*. (Note that the splitting structures here are defined by Lie algebra conditions and rank sizes adjusted to the tautological tuple* $(\underline{B},\underline{\lambda}_B,\underline{i}_B)$ *over* $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$*, using the boundary PEL-type* $\mathcal{O}$*-lattice* $(L^{\mathsf{Z}_{\mathcal{H}}},\langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_{\mathcal{H}}},h_0^{\mathsf{Z}_{\mathcal{H}}})$ *as in* [12, Def. 5.4.2.6]*.)*

Definition 3.2.3. *With the same setting as above, we define* $\vec{\mathsf{M}}^{\Phi_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$*,* $\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$*,* $\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$*,* $\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$*,* $\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$*, and* $\vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ *to be the respective normalizations of the fiber products of* $\vec{\mathsf{M}}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}$*,* $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$*,* $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$*,* $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$*,* $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$*, and* $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ *with* $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$ *over* $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$*, via the canonical structural morphisms.*

Lemma 3.2.4. *We have the following canonical isomorphisms:*

$$\vec{\mathsf{M}}^{\Phi_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}\cong\mathrm{Spl}^{+}_{(\,^{\sharp}\underline{\mathscr{H}},\,^{\sharp}\underline{\mathscr{F}},\,^{\sharp}\underline{j})/(\vec{\mathsf{M}}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}\underset{\mathcal{O}_{F_0,(p)}}{\otimes}\mathcal{O}_K)},\tag{3.2.5}$$

$$\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}\cong\mathrm{Spl}^{+}_{(\,^{\sharp}\underline{\mathscr{H}},\,^{\sharp}\underline{\mathscr{F}},\,^{\sharp}\underline{j})/(\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}\underset{\mathcal{O}_{F_0,(p)}}{\otimes}\mathcal{O}_K)},\tag{3.2.6}$$

$$\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}\cong\mathrm{Spl}^{+}_{(\,^{\sharp}\underline{\mathscr{H}},\,^{\sharp}\underline{\mathscr{F}},\,^{\sharp}\underline{j})/(\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}\underset{\mathcal{O}_{F_0,(p)}}{\otimes}\mathcal{O}_K)},\tag{3.2.7}$$

$$\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)\cong\mathrm{Spl}^{+}_{(\,^{\sharp}\underline{\mathscr{H}},\,^{\sharp}\underline{\mathscr{F}},\,^{\sharp}\underline{j})/(\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)\underset{\mathcal{O}_{F_0,(p)}}{\otimes}\mathcal{O}_K)},\tag{3.2.8}$$

$$\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \cong \mathrm{Spl}^{+}_{(\,^\sharp\underline{\mathscr{H}},\,^\sharp\underline{\mathscr{F}},\,^\sharp\underline{j})/(\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K)}, \tag{3.2.9}$$

and

$$\vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \cong \mathrm{Spl}^{+}_{(\,^\sharp\underline{\mathscr{H}},\,^\sharp\underline{\mathscr{F}},\,^\sharp\underline{j})/(\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K)}, \tag{3.2.10}$$

where

$$\mathrm{Spl}^{+}_{(\,^\sharp\underline{\mathscr{H}},\,^\sharp\underline{\mathscr{F}},\,^\sharp\underline{j})/(\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K)}$$

is a relative scheme over $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K$ *(see* [8]*), which compatibly assigns to affine open formal subschemes* $\mathrm{Spf}(R)$ *of* $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K$ *the corresponding schemes* $\mathrm{Spl}^{+}_{(\,^\sharp\underline{\mathscr{H}},\,^\sharp\underline{\mathscr{F}},\,^\sharp\underline{j})/\,\mathrm{Spec}(R)}$ *over* $\mathrm{Spec}(R)$.

Proof. Since $\mathrm{Spl}_{(\,^\sharp\underline{\mathscr{H}},\,^\sharp\underline{\mathscr{F}},\,^\sharp\underline{j})/(\vec{\mathsf{M}}^{\mathrm{Z}_{\mathcal{H}}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K)}$ represents the functor assigning to each scheme the isomorphism classes of splitting structures for pullbacks of $(\,^\sharp\underline{\mathscr{H}},\,^\sharp\underline{\mathscr{F}},\,^\sharp\underline{j})$ (see Proposition 2.3.7), and since the various objects on the right-hand sides are defined by taking normalizations (see Proposition 2.3.10), these follow from the definitions of the various objects on the left-hand sides (see Definition 3.2.3). □

Proposition 3.2.11 (cf. [12, Prop. 6.2.4.7 and (6.2.4.8); see also the errata], [13, Prop. 8.7, 8.14, and 8.20], and [15, Constr. 4.5])**.** *The canonical morphism* $\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ *is a torsor under the split torus* $E_{\Phi_{\mathcal{H}}}$ *with character group* $\mathbf{S}_{\Phi_{\mathcal{H}}}$*, the canonical morphism* $\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ *is a torsor under the split torus* $E_{\Phi_{\mathcal{H}},\sigma}$ *with character group* $\sigma^{\perp} := \{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}} : \langle \ell, y\rangle = 0\ \forall y \in \sigma\}$ *(see* [12, Def. 6.1.2.5]*), and the canonical morphism* $\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ *over* $\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ *is an open immersion defining an affine toroidal embedding associated with the cone* $\sigma \in \Sigma_{\Phi_{\mathcal{H}}} \in \Sigma$*. Moreover, the canonical morphisms*

$$\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \underset{\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}, \tag{3.2.12}$$

$$\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma) \to \vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma) \underset{\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}, \tag{3.2.13}$$

and

$$\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \underset{\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \tag{3.2.14}$$

are $E_{\Phi_{\mathcal{H}}}$*-equivariant isomorphisms over* $\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$*, which are compatible with each other. Consequently, the pullback of* [13, (8.10)] *gives a canonical homomorphism*

$$\mathbf{S}_{\Phi_{\mathcal{H}}} \to \underline{\mathrm{Pic}}(\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}) : \ell \mapsto \vec{\Psi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell), \tag{3.2.15}$$

giving for each $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}$ *an invertible sheaf* $\vec{\Psi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ *over* $\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ *(up to isomorphism), together with isomorphisms*

$$\vec{\Delta}^{\mathrm{spl},*}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\ell,\ell'} : \vec{\Psi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathscr{O}_{\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \vec{\Psi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell') \xrightarrow{\sim} \vec{\Psi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell + \ell')$$

for all $\ell, \ell' \in \mathbf{S}_{\Phi_\mathcal{H}}$*, satisfying the necessary compatibilities with each other making* $\bigoplus\limits_{\ell \in \mathbf{S}_{\Phi_\mathcal{H}}} \vec{\Psi}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell)$ *an* $\mathscr{O}_{\vec{C}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}}}$*-algebra, such that*

$$\vec{\Xi}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}} \cong \underline{\mathrm{Spec}}_{\mathscr{O}_{\vec{C}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}}}} \Big(\bigoplus_{\ell \in \mathbf{S}_{\Phi_\mathcal{H}}} \vec{\Psi}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell) \Big), \tag{3.2.16}$$

$$\vec{\Xi}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\sigma) \cong \underline{\mathrm{Spec}}_{\mathscr{O}_{\vec{C}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}}}} \Big(\bigoplus_{\ell \in \sigma^\vee} \vec{\Psi}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell) \Big), \tag{3.2.17}$$

where $\sigma^\vee := \{\ell \in \mathbf{S}_{\Phi_\mathcal{H}} : \langle \ell, y \rangle \geq 0 \ \forall y \in \sigma\}$ *as usual* (*see* [12, Def. 6.1.1.8])*, and*

$$\vec{\Xi}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H},\sigma} \cong \underline{\mathrm{Spec}}_{\mathscr{O}_{\vec{C}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}}}} \Big(\bigoplus_{\ell \in \sigma^\perp} \vec{\Psi}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell) \Big). \tag{3.2.18}$$

Proof. These follow from [13, Prop. 8.7, 8.14, and 8.20] and the arguments there, because the pullback of the $E_{\Phi_\mathcal{H}}$-torsor $\vec{\Xi}_{\Phi_\mathcal{H},\delta_\mathcal{H}} \to \vec{C}_{\Phi_\mathcal{H},\delta_\mathcal{H}}$ under $\vec{C}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}} \to \vec{C}_{\Phi_\mathcal{H},\delta_\mathcal{H}}$ is necessarily normal, and hence is isomorphic to $\vec{\Xi}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}}$ via the canonical morphism (3.2.12), not just as a scheme but also as an $E_{\Phi_\mathcal{H}}$-torsor. ☐

Remark 3.2.19 (cf. Remark 2.4.13). To summarize, we have a commutative diagram

$$\begin{array}{ccccc}
\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H},\sigma} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\Xi}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H},\sigma} & \twoheadrightarrow & \vec{\Xi}_{\Phi_\mathcal{H},\delta_\mathcal{H},\sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathfrak{X}_{\Phi_\mathcal{H},\delta_\mathcal{H},\sigma} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H},\sigma} & \twoheadrightarrow & \vec{\mathfrak{X}}_{\Phi_\mathcal{H},\delta_\mathcal{H},\sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\sigma) \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\Xi}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\sigma) & \twoheadrightarrow & \vec{\Xi}_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\sigma) \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \\
\big\uparrow & & \big\uparrow & & \big\uparrow \\
\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\Xi}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}} & \twoheadrightarrow & \vec{\Xi}_{\Phi_\mathcal{H},\delta_\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
C_{\Phi_\mathcal{H},\delta_\mathcal{H}} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{C}^{\mathrm{spl}}_{\Phi_\mathcal{H},\delta_\mathcal{H}} & \twoheadrightarrow & \vec{C}_{\Phi_\mathcal{H},\delta_\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathsf{M}^{\Phi_\mathcal{H}}_\mathcal{H} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathsf{M}}^{\Phi_\mathcal{H},\mathrm{spl}}_\mathcal{H} & \twoheadrightarrow & \vec{\mathsf{M}}^{\Phi_\mathcal{H}}_\mathcal{H} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathsf{M}^{\mathsf{Z}_\mathcal{H}}_\mathcal{H} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathsf{M}}^{\mathsf{Z}_\mathcal{H},\mathrm{spl}}_\mathcal{H} & \twoheadrightarrow & \vec{\mathsf{M}}^{\mathsf{Z}_\mathcal{H}}_\mathcal{H} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K
\end{array} \tag{3.2.20}$$

in which all squares not involving $\vec{\mathsf{M}}_{\mathcal{H}}^{\Phi_{\mathcal{H}}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K$ and $\vec{\mathsf{M}}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K$ are Cartesian. The horizontal arrows at the left-hand sides are open immersions with schematically dense images, because the bottom one is so by definition. The horizontal arrows at the right-hand sides are projective (which are the Spl^+ over the respective bases, as in Proposition 2.3.10) and surjective, whose pre-composition with the horizontal arrows at the left-hand sides in the same rows are still open immersions with schematically dense images. The (vertical) arrows between the top two rows are closed immersions, while the arrows between the second and third rows are formal completions. The arrows between the third and fourth rows are given by affine toroidal embeddings associated with the cone σ. The arrows between the fourth and fifth rows are (smooth) torsors under the same split torus $E_{\Phi_{\mathcal{H}}}$. The arrows between the fifth and sixth rows are all proper and surjective with the left-most one being an abelian scheme torsor. The arrows between the bottom two rows are all finite and surjective with the left-most one being étale. The commutative diagram can be further expanded by adding vertical arrows from the first row to the fifth row, which are (smooth) torsors under the same split torus $E_{\Phi_{\mathcal{H}},\sigma}$.

By [13, Thm. 10.13] and [15, Thm. 6.1(4)], there is a canonical isomorphism

$$(\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \xrightarrow{\sim} \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}. \tag{3.2.21}$$

Lemma 3.2.22. *For each $\Lambda \in \mathscr{L}$, there exist split tori T_Λ and $T_\Lambda^\vee$, with character groups some $\mathcal{O}$-lattices X_Λ and Y_Λ, such that we have short exact sequences*

$$1 \to T_\Lambda \to A_\Lambda^{\mathrm{ext}} \to B_\Lambda \to 1 \tag{3.2.23}$$

and

$$1 \to T_\Lambda^\vee \to A_\Lambda^{\mathrm{ext},\vee} \to B_\Lambda^\vee \to 1 \tag{3.2.24}$$

of (relative) group schemes over $(\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$, where A_Λ^{ext} and $A_\Lambda^{\mathrm{ext},\vee}$ abusively denote (by the same symbols) the pullbacks to $(\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ of the semi-abelian schemes A_Λ^{ext} and $A_\Lambda^{\mathrm{ext},\vee}$ over $\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$, respectively. Moreover, we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_\Lambda & \longrightarrow & A_\Lambda^{\mathrm{ext}} & \longrightarrow & B_\Lambda & \longrightarrow & 1 \\ & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\ 1 & \longrightarrow & T_{\Lambda^\#}^\vee & \longrightarrow & A_{\Lambda^\#}^{\mathrm{ext},\vee} & \longrightarrow & B_{\Lambda^\#}^\vee & \longrightarrow & 1 \end{array} \tag{3.2.25}$$

in which the left-most vertical arrow is dual to a canonical isomorphism $Y_{\Lambda^\#} \xrightarrow{\sim} X_\Lambda$; the middle vertical arrow is the pullback to $(\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ of the unique extension over the noetherian normal scheme $\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ of the isomorphism $A_\Lambda \xrightarrow{\sim} A_{\Lambda^\#}^\vee$ over $\vec{\mathsf{M}}_{\mathcal{H}}$ (see [12, Prop. 3.3.1.5]*), which is part of the data of $(\underline{A},\underline{\lambda},\underline{i})$ over $\vec{\mathsf{M}}_{\mathcal{H}}$; and where the right-most vertical arrow is the pullback to $(\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ of the isomorphism $B_\Lambda \xrightarrow{\sim} B_{\Lambda^\#}^\vee$ over $\vec{\mathsf{M}}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$, which is part of the data of $(\underline{B},\underline{\lambda_B},\underline{i_B})$ over $\vec{\mathsf{M}}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$.*

Proof. First consider the special case where $\Lambda = p^r L_{\mathrm{j}} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$, for some $r \in \mathbb{Z}$ and $\mathrm{j} \in \mathrm{J}$. By the construction of $A_\Lambda^{\mathrm{ext}} = \vec{A}_{\mathrm{j}}^{\mathrm{ext}}$ and $A_\Lambda^{\mathrm{ext},\vee} = \vec{A}_{\mathrm{j}}^{\mathrm{ext},\vee}$ over $\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$, which was based on [13, Lem. 11.1 and Thm. 11.2] and [15, Thm. 6.1] (or more precisely [15, Lem. 5.19 and Prop. 5.20]), their pullbacks to $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ are isomorphic to the pullbacks of the Mumford families ${}^{\heartsuit}\vec{G}_{\mathrm{j}}$ and ${}^{\heartsuit}\vec{G}^{\vee}_{\mathrm{j}}$ over $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (see [12, Def. 6.2.5.28] and [13, (8.29)]), respectively. Then it follows from the constructions of the Mumford families there that we have canonical short exact sequences

$$1 \to T_{\mathrm{j}} \to {}^{\heartsuit}\vec{G}_{\mathrm{j}} \to B_{\mathrm{j}} \to 1 \tag{3.2.26}$$

and

$$1 \to T^{\vee}_{\mathrm{j}} \to {}^{\heartsuit}\vec{G}^{\vee}_{\mathrm{j}} \to B^{\vee}_{\mathrm{j}} \to 1 \tag{3.2.27}$$

for the split tori T_{j} and $T^{\vee}_{\mathrm{j}}$ with character groups X_{j} and Y_{j}, respectively, where X_{j} and Y_{j} are part of the torus argument $\Phi_{\mathcal{H}_{\mathrm{j}}} = (X_{\mathrm{j}}, Y_{\mathrm{j}}, \phi_{\mathrm{j}}, \varphi_{-2,\mathcal{H}_{\mathrm{j}}}, \varphi_{0,\mathcal{H}_{\mathrm{j}}})$ associated with $\Phi_{\mathcal{H}}$ as in [13, (3.8)]. In this special case, T_{j}, $T^{\vee}_{\mathrm{j}}$, (3.2.26), and (3.2.27) give up to (compatible) $\mathbb{Z}^{\times}_{(p)}$-isogenies the T_{Λ}, $T^{\vee}_{\Lambda}$, (3.2.23), and (3.2.24) we want. For general $\Lambda \in \mathscr{L}$, as in the proof of Lemma 3.1.1, we have an isogeny $\vec{A}^{\mathrm{ext}}_{\mathrm{j}} \to A^{\mathrm{ext}}_{\Lambda}$ of semi-abelian schemes over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, for some $\mathrm{j} \in \mathrm{J}$, which induces isogenies of Raynaud extensions and of dual Raynaud extensions, by the constructions in [12, Sec. 3.3.3, 3.4.1, and 3.4.4], which give the desired T_{Λ}, $T^{\vee}_{\Lambda}$, (3.2.23), and (3.2.24) over $(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$. As for the commutative diagram (3.2.25), it suffices to note that, in the proof of Proposition 2.2.11, the polarization $\lambda_{\Lambda_0} : A_{\Lambda_0} \to A^{\vee}_{\Lambda_0}$ in Lemma 2.2.2 is defined to be $(f^{\vee}_{p^{-r_0}\Lambda_{\mathrm{j}_0},\Lambda_0})^{-1} \circ \vec{\lambda}_{\mathrm{j}_0} \circ f^{-1}_{p^{-r_0}\Lambda_{\mathrm{j}_0},\Lambda_0}$ over $\vec{\mathsf{M}}_{\mathcal{H}}$, for any $\mathrm{j}_0 \in \mathscr{L}$ (satisfying $\Lambda_{\mathrm{j}_0} \subset p^{r_0}\Lambda_0$ as in Choices 2.2.9), which (uniquely) extends to $(f^{\mathrm{ext},\vee}_{p^{-r_0}\Lambda_{\mathrm{j}_0},\Lambda_0})^{-1} \circ \vec{\lambda}^{\mathrm{ext}}_{\mathrm{j}_0} \circ (f^{\mathrm{ext}}_{p^{-r_0}\Lambda_{\mathrm{j}_0},\Lambda_0})^{-1}$ (with the superscript "ext" denoting the unique extensions of homomorphisms of semi-abelian schemes) over the noetherian normal scheme $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (by [12, Prop. 3.3.1.5]), and we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_{\mathrm{j}_0} & \longrightarrow & {}^{\heartsuit}\vec{G}_{\mathrm{j}_0} & \longrightarrow & B_{\mathrm{j}_0} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \lambda_{T_{\mathrm{j}_0}}} & & \downarrow{\scriptstyle {}^{\heartsuit}\vec{\lambda}_{\mathrm{j}_0}} & & \downarrow{\scriptstyle \lambda_{B_{\mathrm{j}_0}}} & & \\ 1 & \longrightarrow & T^{\vee}_{\mathrm{j}_0} & \longrightarrow & {}^{\heartsuit}\vec{G}^{\vee}_{\mathrm{j}_0} & \longrightarrow & B^{\vee}_{\mathrm{j}_0} & \longrightarrow & 1 \end{array} \tag{3.2.28}$$

canonically associated with the Mumford family $({}^{\heartsuit}\vec{G}_{\mathrm{j}_0}, {}^{\heartsuit}\vec{\lambda}_{\mathrm{j}_0}, {}^{\heartsuit}\vec{i}_{\mathrm{j}_0}, {}^{\heartsuit}\vec{\alpha}_{\mathcal{H}_{\mathrm{j}_0}})$, which induces (3.2.25) for all other $\Lambda \in \mathscr{L}$ by using the $\mathbb{Q}^{\times}$-isogenies $f^{\mathrm{ext}}_{\Lambda',\Lambda''} : A^{\mathrm{ext}}_{\Lambda'} \to A^{\mathrm{ext}}_{\Lambda''}$ associated with all the inclusions $\Lambda' \subset \Lambda''$ in $\mathscr{L}$ (see Lemma 3.1.1). □

3.3. **Comparison of formal completions.**

Theorem 3.3.1. *There is a canonical isomorphism*

$$(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \xrightarrow{\sim} \vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}, \tag{3.3.2}$$

where $(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ *is defined as in the end of Section* 3.1*, covering the canonical isomorphism* (3.2.21)*. Then* (3.3.2) *induces a canonical isomorphism*

$$\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \xrightarrow{\sim} \vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \tag{3.3.3}$$

covering the isomorphism $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \xrightarrow{\sim} \vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ *(see* [13, Cor. 10.15] *and* [15, Thm. 6.1(5)]*).*

Remark 3.3.4. Since both sides of (3.3.2) are separated and have schematically dense characteristic zero fibers isomorphic to those of (3.2.21) by Lemma 2.3.9, the condition that (3.3.2) covers (3.2.21) forces (3.3.2) to be unique if it exists. Any isomorphism as in (3.3.2) then canonically induces an isomorphism as in (3.3.3).

The remainder of this section will be devoted to the proof of Theorem 3.3.1. By Remark 3.3.4, it suffices to construct an isomorphism (3.3.2) covering (3.2.21).

For simplicity of notation, in the remainder of this section, let us write

$$\mathfrak{X} := (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \tag{3.3.5}$$

and

$$\mathfrak{X}^{\mathrm{spl}} := (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}. \tag{3.3.6}$$

As in Definition 3.2.1, let us denote by the same symbols the pullback to $\mathfrak{X}$ of the $({}^{\sharp}\underline{\mathscr{H}}, {}^{\sharp}\underline{\mathscr{F}}, {}^{\sharp}\underline{j})$ over $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ under the composition $\mathfrak{X} \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ of (3.2.21) and the structural morphism $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ (see (3.2.20)); and let us denote by $({}^{\natural}\underline{\mathscr{H}}, {}^{\natural}\underline{\mathscr{F}}, {}^{\natural}\underline{j})$ the pullback to $\mathfrak{X}$ of the $(\underline{\mathscr{H}}^{\mathrm{ext}}, \underline{\mathscr{F}}^{\mathrm{ext}}, \underline{j}^{\mathrm{ext}})$ as in Proposition 3.1.2 under the canonical morphism $\mathfrak{X} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. Then we can abusively write

$$\mathfrak{X}^{\mathrm{spl}} \cong \mathrm{Spl}^{+}_{({}^{\natural}\underline{\mathscr{H}}, {}^{\natural}\underline{\mathscr{F}}, {}^{\natural}\underline{j})/\mathfrak{X}} \tag{3.3.7}$$

(cf. Proposition 2.3.10 and Remark 3.2.19) and

$$\vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \cong \mathrm{Spl}^{+}_{({}^{\sharp}\underline{\mathscr{H}}, {}^{\sharp}\underline{\mathscr{F}}, {}^{\sharp}\underline{j})/\mathfrak{X}} \tag{3.3.8}$$

(cf. (3.2.10)), where the right-hand sides of (3.3.7) and (3.3.8) are relative schemes over $\mathfrak{X}$ (see [8]; cf. the explanation in Lemma 3.2.4).

Lemma 3.3.9. *For all $\Lambda \in \mathscr{L}$ and $[\tau] \in \Upsilon/\sim$, we have canonical short exact sequences*

$$0 \to {}^{\sharp}\mathscr{F}_{\Lambda,[\tau]} \to {}^{\natural}\mathscr{F}_{\Lambda,[\tau]} \to {}^{\flat}\mathscr{F}_{\Lambda,[\tau]} \to 0 \tag{3.3.10}$$

of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathfrak{X}}$-modules, where ${}^{\sharp}\mathscr{F}_{\Lambda,[\tau]}$, ${}^{\natural}\mathscr{F}_{\Lambda,[\tau]}$, and ${}^{\flat}\mathscr{F}_{\Lambda,[\tau]}$ can be identified with the $\mathscr{O}_{\mathfrak{X}}$-module local direct summands $(\underline{\mathrm{Lie}}^{\vee}_{B^{\vee}_{\Lambda}/\mathfrak{X}})_{[\tau]}$, $(\underline{\mathrm{Lie}}^{\vee}_{A^{\mathrm{ext},\vee}_{\Lambda}/\mathfrak{X}})_{[\tau]}$, and $(\underline{\mathrm{Lie}}^{\vee}_{T^{\vee}_{\Lambda}/\mathfrak{X}})_{[\tau]}$ of $\underline{\mathrm{Lie}}^{\vee}_{B^{\vee}_{\Lambda}/\mathfrak{X}}$, $\underline{\mathrm{Lie}}^{\vee}_{A^{\mathrm{ext},\vee}_{\Lambda}/\mathfrak{X}}$, and $\underline{\mathrm{Lie}}^{\vee}_{T^{\vee}_{\Lambda}/\mathfrak{X}}$, respectively, defined as in (2.1.5).

Proof. Since $\mathscr{F}^{\mathrm{ext}}_{\Lambda} \cong \underline{\mathrm{Lie}}^{\vee}_{A^{\mathrm{ext},\vee}_{\Lambda}/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$, this follows from the short exact sequence for duals of relative Lie algebras induced by (3.2.24). □

Lemma 3.3.11. *Consider any object*

$$\{({}^{\natural}\underline{\mathscr{F}}^{i}_{[\tau]}, {}^{\natural}\underline{j}^{i}_{[\tau]})\}_{[\tau]\in\Upsilon/\sim, 0\leq i<d_{[\tau]}}$$

parameterized by $\mathrm{Spl}'_{({}^{\natural}\underline{\mathscr{H}}, {}^{\natural}\underline{\mathscr{F}}, {}^{\natural}\underline{j})/\mathfrak{X}}$ *(cf.* (3.3.7)*), without condition* (4) *in Definition* 2.3.3*. For all $\Lambda \in \mathscr{L}$ and $[\tau] \in \Upsilon/\sim$, and for all integers i satisfying $0 \leq i < d_{[\tau]}$, let*

$${}^{\sharp}\mathscr{F}^{i}_{\Lambda,[\tau]} := {}^{\natural}\mathscr{F}^{i}_{\Lambda,[\tau]} \cap {}^{\sharp}\mathscr{F}_{\Lambda,[\tau]} \tag{3.3.12}$$

and

$${}^{\flat}\mathscr{F}^{i}_{\Lambda,[\tau]} := {}^{\natural}\mathscr{F}^{i}_{\Lambda,[\tau]} / {}^{\sharp}\mathscr{F}^{i}_{\Lambda,[\tau]}. \tag{3.3.13}$$

Then the graded pieces ${}^{\sharp}\mathscr{F}^i_{\Lambda,[\tau]}/{}^{\sharp}\mathscr{F}^{i+1}_{\Lambda,[\tau]}$ *and* ${}^{\flat}\mathscr{F}^i_{\Lambda,[\tau]}/{}^{\flat}\mathscr{F}^{i+1}_{\Lambda,[\tau]}$ *are annihilated by* $b\otimes 1-1\otimes\tau_{[\tau],i}(b)$ *for all* $b\in\mathcal{O}_{F_{[\tau]}}$*. Moreover, for each unit* b *of* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}_p$ *which normalizes* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_p$*, the periodicity isomorphism* $\theta^b_{{}^{\natural}\mathscr{F}^i_{\Lambda,[\tau]}}$ *induces the* **periodicity isomorphisms** $\theta^b_{{}^{\sharp}\mathscr{F}^i_{\Lambda,[\tau]}}:({}^{\sharp}\mathscr{F}^i_{\Lambda,[\tau]})^b\overset{\sim}{\to}{}^{\sharp}\mathscr{F}^i_{b\Lambda,[\tau]}$ *and* $\theta^b_{{}^{\flat}\mathscr{F}^i_{\Lambda,[\tau]}}:({}^{\flat}\mathscr{F}^i_{\Lambda,[\tau]})^b\overset{\sim}{\to}{}^{\flat}\mathscr{F}^i_{b\Lambda,[\tau]}$ *of* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathscr{O}_{\mathfrak{X}}$*-modules.*

Proof. Since (3.3.10) is an exact sequence of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathscr{O}_{\mathfrak{X}}$-modules, these follow from the very definitions of ${}^{\sharp}\mathscr{F}^i_{\Lambda,[\tau]}$ and ${}^{\flat}\mathscr{F}^i_{\Lambda,[\tau]}$. □

Lemma 3.3.14. *Let* M *be any* $\mathcal{O}_{F_{[\tau]}}$*-lattice, let* S *be any scheme or formal scheme over* $\mathrm{Spec}(\mathcal{O}_K)$*, and let* $\mathscr{M}:=M\underset{\mathbb{Z}_p}{\otimes}\mathscr{O}_S$*. Then there exists a unique filtration*

$$0=\mathscr{M}^{d_{[\tau]}}\subset\mathscr{M}^{d_{[\tau]}-1}\subset\cdots\subset\mathscr{M}^1\subset\mathscr{M}^0=\mathscr{M} \tag{3.3.15}$$

of coherent $\mathscr{O}_S$*-submodules of* $\mathscr{M}$ *such that, for each integer* i *satisfying* $0\leq i<d_{[\tau]}$*, the quotient* $\mathscr{M}^i/\mathscr{M}^{i+1}$ *is annihilated by* $b\otimes 1-1\otimes\tau_{[\tau],i}(b)$ *for all* $b\in\mathcal{O}_{F_{[\tau]}}$*. The graded pieces* $\mathscr{M}^i/\mathscr{M}^{i+1}$ *are automatically locally free* $\mathscr{O}_S$*-modules of finite rank, and hence both* $\mathscr{M}^i$ *and* $\mathscr{M}/\mathscr{M}^i$ *are locally free* $\mathscr{O}_S$*-modules of finite rank. Moreover,* $\mathscr{M}^i$ *is the* $\mathscr{O}_S$*-submodule of* $\mathscr{M}$ *spanned by the images of the endomorphism* $\prod\limits_{0\leq k<i}(b_k\otimes 1-1\otimes\tau_{[\tau],k}(b_k))$ *of* $\mathscr{M}$*, for all elements* $b_0,b_1,\ldots,b_{i-1}\in\mathcal{O}_{F_{[\tau]}}$*; it is also the intersection of the kernels of the endomorphisms* $\prod\limits_{i\leq k<d_{[\tau]}}(b_k\otimes 1-1\otimes\tau_{[\tau],k}(b_k))$ *of* $\mathscr{M}$*, for all elements* $b_i,b_{i+1},\ldots,b_{d_{[\tau]}-1}\in\mathcal{O}_{F_{[\tau]}}$*.*

Proof. Let K_0 denote the maximal unramified extension of $\mathbb{Q}_p$ in K, so that $F_{[\tau]}\underset{\mathbb{Q}_p}{\otimes}K_0\cong\prod\limits_{\alpha}F_\alpha$ for some totally ramified field extensions F_α of K_0. Since $\mathcal{O}_{K_0}$ is finite étale over $\mathbb{Z}_p$, the canonical morphism $\mathcal{O}_{F_{[\tau]}}\underset{\mathbb{Z}_p}{\otimes}\mathcal{O}_{K_0}\to\prod\limits_{\alpha}\mathcal{O}_{F_\alpha}$ is an isomorphism, because both sides are normal and have the same total ring of fractions $F_{[\tau]}\underset{\mathbb{Q}_p}{\otimes}K_0$. Accordingly, the $\mathcal{O}_{F_{[\tau]}}\underset{\mathbb{Z}_p}{\otimes}\mathcal{O}_{K_0}$-module $M\underset{\mathbb{Z}_p}{\otimes}\mathcal{O}_{K_0}$ and the sheaf $\mathscr{M}\cong(M\underset{\mathbb{Z}_p}{\otimes}\mathcal{O}_{K_0})\underset{\mathcal{O}_{K_0}}{\otimes}\mathscr{O}_S$ compatibly decompose into direct sums, where $\mathcal{O}_{F_{[\tau]}}\underset{\mathbb{Z}_p}{\otimes}\mathcal{O}_{K_0}$ acts on each summand via some factor $\mathcal{O}_{F_\alpha}$. Thus, in order to prove the lemma, we may and we shall replace $\mathcal{O}_{F_{[\tau]}}$ with some factor $\mathcal{O}_{F_\alpha}$, and replace M with the corresponding summand of $M\underset{\mathbb{Z}_p}{\otimes}\mathcal{O}_{K_0}$. Now that F_α is a totally ramified (separable) extension of K_0, the lemma follows by writing each $\mathscr{M}^i$ as both the image of some $Q^i(T)$ and the kernel of some $Q_i(T)$ as in [17, (2.4)], whose formation is compatible with arbitrary base changes, and hence must be $\mathscr{O}_S$-module local direct summands of $\mathscr{M}$, as desired. □

Corollary 3.3.16. *The sub-*$\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathscr{O}_{\mathfrak{X}}$*-modules* ${}^{\flat}\mathscr{F}^i_{\Lambda,[\tau]}$ *of* ${}^{\flat}\mathscr{F}_{\Lambda,[\tau]}$ *in Lemma* 3.3.11 *are locally free and independent of the filtrations* $\{{}^{\natural}\mathscr{F}^i_{\Lambda,[\tau]}\}_{0\leq i<d_{[\tau]}}$ *on* ${}^{\natural}\mathscr{F}_{\Lambda,[\tau]}$*.*

Proof. By Lemma 3.2.22, the character group of the split torus $T^\vee_\Lambda$ is an $\mathcal{O}$-lattice Y_Λ, and so $\underline{\mathrm{Lie}}^\vee_{T^\vee_\Lambda/\mathfrak{X}}\cong Y_\Lambda\underset{\mathbb{Z}}{\otimes}\mathscr{O}_{\mathfrak{X}}\cong(Y_\Lambda\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_p)\underset{\mathbb{Z}_p}{\otimes}\mathscr{O}_{\mathfrak{X}}$, where the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_p$-lattice $Y_\Lambda\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_p$ is an $\mathcal{O}_F\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_p$-lattice because $\mathcal{O}$ is maximal at p, by Assumption 2.1.1. Let us write

$Y_\Lambda \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p \cong \underset{[\tau]\in\Upsilon/\sim}{\oplus} Y_{\Lambda,[\tau]}$ as in (2.1.5). Then ${}^\flat\mathscr{F}_{\Lambda,[\tau]} \cong (\underline{\mathrm{Lie}}^\vee_{T^\vee_\Lambda/\mathfrak{X}})_{[\tau]} \cong Y_{\Lambda,[\tau]} \underset{\mathbb{Z}_p}{\otimes} \mathscr{O}_\mathfrak{X}$ by Lemma 3.3.9, and the corollary follows from Lemma 3.3.14, as desired. □

Proposition 3.3.17. *The sub-$\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_\mathfrak{X}$-modules ${}^\sharp\mathscr{F}^i_{\Lambda,[\tau]}$ of ${}^\sharp\mathscr{F}_{\Lambda,[\tau]}$ in Lemma* 3.3.11 *are locally free $\mathscr{O}_\mathfrak{X}$-modules. Together with the canonical embeddings*

$$ {}^\sharp j^i_{\Lambda,[\tau]} : {}^\sharp\mathscr{F}^i_{\Lambda,[\tau]} \to {}^\sharp\mathscr{H}_{\Lambda,[\tau]} $$

defined by composing the canonical embeddings ${}^\sharp\mathscr{F}^i_{\Lambda,[\tau]} \to {}^\sharp\mathscr{F}_{\Lambda,[\tau]}$ and ${}^\sharp\mathscr{F}_{\Lambda,[\tau]} \to {}^\sharp\mathscr{H}_{\Lambda,[\tau]}$, we obtain a splitting structure

$$ \{({}^\sharp\underline{\mathscr{F}}^i_{[\tau]}, {}^\sharp\underline{j}^i_{[\tau]})\}_{[\tau]\in\Upsilon/\sim, 0\le i<d_{[\tau]}} $$

for $({}^\sharp\underline{\mathscr{H}}, {}^\sharp\underline{\mathscr{F}}, {}^\sharp\underline{j})$ over $\mathfrak{X}$, parameterized by $\mathrm{Spl}^+_{({}^\sharp\underline{\mathscr{H}}, {}^\sharp\underline{\mathscr{F}}, {}^\sharp\underline{j})/\mathfrak{X}}$. By repeating the same construction over affine formal schemes over $\mathfrak{X}$, we obtain a canonical morphism

$$ \mathrm{Spl}'_{({}^\natural\underline{\mathscr{H}}, {}^\natural\underline{\mathscr{F}}, {}^\natural\underline{j})/\mathfrak{X}} \to \mathrm{Spl}'_{({}^\sharp\underline{\mathscr{H}}, {}^\sharp\underline{\mathscr{F}}, {}^\sharp\underline{j})/\mathfrak{X}} \tag{3.3.18} $$

over $\mathfrak{X}$, which induces a canonical morphism

$$ \mathrm{Spl}^+_{({}^\natural\underline{\mathscr{H}}, {}^\natural\underline{\mathscr{F}}, {}^\natural\underline{j})/\mathfrak{X}} \to \mathrm{Spl}^+_{({}^\sharp\underline{\mathscr{H}}, {}^\sharp\underline{\mathscr{F}}, {}^\sharp\underline{j})/\mathfrak{X}} \tag{3.3.19} $$

over $\mathfrak{X}$, by Lemma 2.3.9 *and by the second paragraph of Proposition* 2.3.10.

Proof. Since we have a short exact sequence

$$ 0 \to {}^\sharp\mathscr{F}^i_{\Lambda,[\tau]} \to {}^\natural\mathscr{F}^i_{\Lambda,[\tau]} \to {}^\flat\mathscr{F}^i_{\Lambda,[\tau]} \to 0 \tag{3.3.20} $$

by definition (see Lemma 3.3.11), and since ${}^\natural\mathscr{F}^i_{\Lambda,[\tau]}$ and ${}^\flat\mathscr{F}^i_{\Lambda,[\tau]}$ are locally free $\mathscr{O}_\mathfrak{X}$-modules by definition and by Corollary 3.3.16, ${}^\sharp\mathscr{F}^i_{\Lambda,[\tau]}$ is also a locally free $\mathscr{O}_\mathfrak{X}$-module. Hence, by Lemma 3.3.11, the collection $\{({}^\sharp\underline{\mathscr{F}}^i_{[\tau]}, {}^\sharp\underline{j}^i_{[\tau]})\}_{[\tau]\in\Upsilon/\sim, 0\le i<d_{[\tau]}}$ satisfies all but the last condition (4) in Definition 2.3.3 as a splitting structure for $({}^\sharp\underline{\mathscr{H}}, {}^\sharp\underline{\mathscr{F}}, {}^\sharp\underline{j})$, and defines an object parameterized by the $\mathrm{Spl}'_{({}^\sharp\underline{\mathscr{H}}, {}^\sharp\underline{\mathscr{F}}, {}^\sharp\underline{j})/\mathfrak{X}}$ as in Proposition 2.3.10. Since the same construction works for splittings structures of pullbacks of $({}^\sharp\underline{\mathscr{H}}, {}^\sharp\underline{\mathscr{F}}, {}^\sharp\underline{j})$ to any affine formal schemes over $\mathfrak{X}$, we obtain the canonical morphism (3.3.18) over $\mathfrak{X}$, as desired. □

Proposition 3.3.21. *The canonical morphism* (3.3.19) *is an isomorphism.*

Proof. By Zariski's main theorem (see [7, III-1, 4.4.3, 4.4.11]) and by [7, IV-4, 18.12.6], it suffices to show that the morphism (3.3.18) is a monomorphism. Hence, it suffices to show that, for each affine formal scheme $\mathrm{Spf}(R)$ over $\mathfrak{X}$ such that R is noetherian and local, the induced morphism

$$ \mathrm{Spl}'_{({}^\natural\underline{\mathscr{H}}, {}^\natural\underline{\mathscr{F}}, {}^\natural\underline{j})/\,\mathrm{Spec}(R)} \to \mathrm{Spl}'_{({}^\sharp\underline{\mathscr{H}}, {}^\sharp\underline{\mathscr{F}}, {}^\sharp\underline{j})/\,\mathrm{Spec}(R)} \tag{3.3.22} $$

induces an injection between points over R.

For each $\Lambda \in \mathscr{L}$, since $\underline{\mathrm{Lie}}^\vee_{T^\vee_\Lambda/\,\mathrm{Spec}(R)} \cong (Y_\Lambda \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p) \underset{\mathbb{Z}_p}{\otimes} R$ as in the proof of Corollary 3.3.16, $\underline{\mathrm{Lie}}^\vee_{T^\vee_\Lambda/\,\mathrm{Spec}(R)}$ is a projective $\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} R$-module. Since R is noetherian and local, for all $\Lambda \in \mathscr{L}$ and $[\tau] \in \Upsilon/\sim$, there are (noncanonical) splittings

$$ {}^\natural\mathscr{F}_{\Lambda,[\tau]} \xrightarrow{\sim} {}^\sharp\mathscr{F}_{\Lambda,[\tau]} \oplus {}^\flat\mathscr{F}_{\Lambda,[\tau]} \tag{3.3.23} $$

of the short exact sequences (3.3.10) of $\mathcal{O}_{F_{[\tau]}} \underset{\mathbb{Z}}{\otimes} R$-modules.

Suppose that the filtration $\{{}^\natural\mathscr{F}^i_{\Lambda,[\tau]}\}_{0\leq i<d_{[\tau]}}$ on ${}^\natural\mathscr{F}_{\Lambda,[\tau]}$ induces the filtrations $\{{}^\sharp\mathscr{F}^i_{\Lambda,[\tau]}\}_{0\leq i<d_{[\tau]}}$ and $\{{}^\flat\mathscr{F}^i_{\Lambda,[\tau]}\}_{0\leq i<d_{[\tau]}}$ on ${}^\sharp\mathscr{F}_{\Lambda,[\tau]}$ and ${}^\flat\mathscr{F}_{\Lambda,[\tau]}$, respectively, by the assignments as in (3.3.12) and (3.3.13). By the last assertion in Lemma 3.3.14, ${}^\flat\mathscr{F}^i_{\Lambda,[\tau]}$ is the R-submodule of ${}^\flat\mathscr{F}_{\Lambda,[\tau]}$ spanned by the images of the endomorphisms $\prod\limits_{0\leq k<i}(b_k\otimes 1-1\otimes\tau_{[\tau],k}(b_k))$, for all elements $b_0,b_1,\ldots,b_{i-1}\in\mathcal{O}_{F_{[\tau]}}$. Since the splitting (3.3.23) is $\mathcal{O}_{F_{[\tau]}}\underset{\mathbb{Z}}{\otimes}R$-equivariant, by condition (2) of Definition 2.3.3, it canonically induces a splitting ${}^\natural\mathscr{F}^i_{\Lambda,[\tau]}\xrightarrow{\sim}{}^\sharp\mathscr{F}^i_{\Lambda,[\tau]}\oplus{}^\flat\mathscr{F}^i_{\Lambda,[\tau]}$. Hence, by Corollary 3.3.16, ${}^\natural\mathscr{F}^i_{\Lambda,[\tau]}$ is uniquely determined by ${}^\sharp\mathscr{F}^i_{\Lambda,[\tau]}$. Since this holds for all $[\tau]$ and i, the morphism (3.3.22) induces an injection between points over R, as desired. □

Proof of Theorem 3.3.1. By Remark 3.3.4, it suffices to take (3.3.2) to be the composition of the isomorphisms (3.3.7), (3.3.19), and the inverse of (3.3.8). □

3.4. **Main theorem for toroidal compactifications.**

Theorem 3.4.1 (cf. [12, Thm. 6.4.1.1]). *For each $\mathcal{H}$ as in Choices* 2.2.10*, and for each compatible collection $\Sigma=\{\Sigma_{\Phi_\mathcal{H}}\}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}$ of admissible rational polyhedral cone decomposition data that is projective as in* [15, Def. 2.1 and 2.7] (*satisfying* [12, Cond. 6.2.5.25] *by assumption; which includes the ones induced by auxiliary choices as in* [13, Sec. 7]*, as explained in* [15, Rem. 2.3 and 2.9])*, there is a normal scheme $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ projective and flat over* $\mathrm{Spec}(\mathcal{O}_K)$*, containing the scheme $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ in Definition* 2.4.5 *as an open fiberwise dense subscheme, together with:*

- *a tautological degenerating family*
$$(\vec{G}_\mathrm{j},\vec{\lambda}_\mathrm{j},\vec{i}_\mathrm{j},\vec{\alpha}_{\mathcal{H}_\mathrm{j}})$$
of type $\mathsf{M}_{\mathcal{H}_\mathrm{j}}$ over $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ (*see* [12, Def. 5.3.2.1])*, for each $\mathrm{j}\in\mathrm{J}$, where $\vec{\alpha}_{\mathcal{H}_\mathrm{j}}$ is defined only over the open dense subscheme $\mathsf{M}_\mathcal{H}\underset{F_0}{\otimes}K$ of $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$;*
- *a tautological $\mathscr{L}$-set*
$$(\underline{\mathscr{H}}^{\mathrm{ext}},\underline{\mathscr{F}}^{\mathrm{ext}},\underline{j}^{\mathrm{ext}})$$
of polarized $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}}$-modules extending the $\mathscr{L}$-set $(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})$ of polarized $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}}$-modules associated with the tautological $(\underline{A},\underline{\lambda},\underline{i},\underline{\alpha}_{\mathcal{H}^p})$ over $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ (*see Definition* 2.4.1 *and Lemma* 2.2.7) *that induces compatible isomorphisms $\mathscr{F}^{\mathrm{ext}}_\Lambda\cong\underline{\mathrm{Lie}}^\vee_{A^{\mathrm{ext},\vee}_\Lambda/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ and $\mathscr{H}^{\mathrm{ext}}_\Lambda/\mathscr{F}^{\mathrm{ext}}_\Lambda\cong\underline{\mathrm{Lie}}_{A^{\mathrm{ext}}_\Lambda/\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ extending the canonical isomorphisms $\mathscr{F}_\Lambda\cong\underline{\mathrm{Lie}}^\vee_{A^\vee_\Lambda/\vec{\mathsf{M}}_\mathcal{H}}$ and $\mathscr{H}_\Lambda/\mathscr{F}_\Lambda\cong\underline{\mathrm{Lie}}_{A_\Lambda/\vec{\mathsf{M}}_\mathcal{H}}$, respectively, for all $\Lambda\in\mathscr{L}$* (*see Proposition* 3.1.2)*; and*
- *a tautological splitting structure*
$$\{(\underline{\mathscr{F}}^{\mathrm{ext},i}_{[\tau]},\underline{j}^{\mathrm{ext},i}_{[\tau]})\}_{[\tau]\in\Upsilon/\sim,0\leq i<d_{[\tau]}}$$
for $(\underline{\mathscr{H}}^{\mathrm{ext}},\underline{\mathscr{F}}^{\mathrm{ext}},\underline{j}^{\mathrm{ext}})$ over $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$, which extends the tautological splitting structure $\{(\underline{\mathscr{F}}^i_{[\tau]},\underline{j}^i_{[\tau]})\}_{[\tau]\in\Upsilon/\sim,0\leq i<d_{[\tau]}}$ for $(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})$ over $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ (*see Definition* 2.4.1)*;*

such that we have the following:

(1) *We have a commutative diagram*

$$
\begin{array}{ccccc}
\mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} & \twoheadrightarrow & \vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} & \twoheadrightarrow & \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K
\end{array}
\tag{3.4.2}
$$

of noetherian normal schemes flat over $\mathrm{Spec}(\mathcal{O}_K)$ *and of canonical morphisms (over* $\mathrm{Spec}(\mathcal{O}_K)$*), in which all squares are Cartesian, all vertical arrows are open immersions with fiberwise dense image over* $\mathrm{Spec}(\mathcal{O}_K)$*, the two horizontal arrows at the left-hand side are open immersions with schematically dense images, the two horizontal arrows at the right-hand side are projective and surjective, and the compositions of horizontal arrows in the same rows are also open immersions with schematically dense images.*

(2) $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ *has a stratification by locally closed subschemes*

$$
\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]},
\tag{3.4.3}
$$

with $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ *running through a complete set of equivalence classes of* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ *(as in* [12, Def. 6.2.6.1]*) with* $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ *and* $\sigma \in \Sigma_{\Phi_{\mathcal{H}}} \in \Sigma$ *(see* (3.1.9)*). (Here* $\mathsf{Z}_{\mathcal{H}}$ *is suppressed in the notation by* [12, Conv. 5.4.2.5]*.) In this stratification, the* $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$*-stratum* $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}$ *is contained in the closure of the* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *if and only if* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ *is a face of* $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$ *as in* [12, Thm. 6.3.2.14 and Rem. 6.3.2.15]*. The analogous assertion holds after pullback to fibers over* $\mathrm{Spec}(\mathcal{O}_K)$*.*

The $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$*-stratum* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *is flat over* $\mathrm{Spec}(\mathcal{O}_K)$ *and normal, and is isomorphic to the support of the formal scheme* $\vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ *for any representative* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ *of* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$*. The formal scheme* $\vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ *admits a canonical structure as the completion of an affine toroidal embedding* $\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ *(along its* σ*-stratum* $\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$*) of a torus torsor* $\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ *over a normal scheme* $\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ *flat over* $\mathrm{Spec}(\mathcal{O}_K)$*. The scheme* $\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ *is proper (and surjective) over a finite cover* $\vec{\mathsf{M}}^{\Phi_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$ *of the boundary version* $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$ *of* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ *(cf. Definitions* 2.4.5 *and* 3.2.1*, and the summary in Remark* 3.2.19*). (Note that* $\mathsf{Z}_{\mathcal{H}}$ *and the isomorphism class of* $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$ *depend only on the cusp label* $[(\mathsf{Z}_{\mathcal{H}},\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$*, but not on the choice of the representative* $(\mathsf{Z}_{\mathcal{H}},\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$*.)*

In particular, $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} = \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(0,0,\{0\})]}$ *is an open fiberwise dense stratum in this stratification.*

The stratification (3.4.3) *is compatible with the stratification of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *as in* [13, Thm. 9.13] *and* [15, Thm. 6.1(3)]*, and we have a commutative* *diagram*

$$\begin{array}{ccccc}
\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}\underset{F_0}{\otimes}K & \hookrightarrow & \vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} & \twoheadrightarrow & \vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}\underset{\mathcal{O}_{F_0,(p)}}{\otimes}\mathcal{O}_K \\
\wr\downarrow & & \wr\downarrow & & \wr\downarrow \\
\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}\underset{F_0}{\otimes}K & \hookrightarrow & \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} & \twoheadrightarrow & \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}\underset{\mathcal{O}_{F_0,(p)}}{\otimes}\mathcal{O}_K \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}\underset{F_0}{\otimes}K & \hookrightarrow & \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} & \twoheadrightarrow & \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}\underset{\mathcal{O}_{F_0,(p)}}{\otimes}\mathcal{O}_K
\end{array} \tag{3.4.4}$$

of canonical morphisms, in which all squares not involving $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}\underset{\mathcal{O}_{F_0,(p)}}{\otimes}\mathcal{O}_K$ *are Cartesian, the vertical arrows in the upper-half are isomorphisms, the vertical arrows in the bottom-half are locally closed immersions, the horizontal arrows at the left-hand sides are open immersions with schematically dense images, the horizontal arrows at the right-hand sides are projective and surjective, and the compositions of horizontal arrows in the same rows are also open immersions with schematically dense images.*

(3) *The formal completion* $(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ *of the scheme* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ *along its* (*locally closed*) $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$*-stratum* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *is canonically isomorphic to the formal scheme* $\vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ *for any representative* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ *of* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$. (*See the isomorphism* (3.3.2) *in Theorem* 3.3.1.)

For any open immersion $\mathrm{Spf}(R,I)\to\vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ *inducing morphisms* $\mathrm{Spec}(R)\to\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ *and* $\mathrm{Spec}(R)\to\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ (*via the structural morphisms and the inverse of the above-mentioned isomorphism* (3.3.2)), *the preimage of* $\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ *under* $\mathrm{Spec}(R)\to\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ *coincides with the preimage of* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ *under* $\mathrm{Spec}(R)\to\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$.

For each $\mathrm{j}\in\mathrm{J}$, *the pullback to* $(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ *of the degenerating family* $(\vec{G}_{\mathrm{j}},\vec{\lambda}_{\mathrm{j}},\vec{i}_{\mathrm{j}},\vec{\alpha}_{\mathcal{H}_{\mathrm{j}}})$ *over* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ *is canonically isomorphic to the pullback to* $\vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ *of the Mumford family* $({}^{\heartsuit}\vec{G}_{\mathrm{j}},\ {}^{\heartsuit}\vec{\lambda}_{\mathrm{j}},\ {}^{\heartsuit}\vec{i}_{\mathrm{j}},\ {}^{\heartsuit}\vec{\alpha}_{\mathcal{H}_{\mathrm{j}}})$ *over* $\vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (*see* [12, Def. 6.2.5.28] *and* [13, (8.29)]), *after we identify the bases using the above-mentioned canonical isomorphism* (3.3.2).

Then we have a commutative diagram

(3.4.5)

$$\begin{array}{ccccc}
\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} & \twoheadrightarrow & \vec{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \\
\wr\downarrow & & \wr\downarrow & & \wr\downarrow \\
(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \underset{F_0}{\otimes} K & \hookrightarrow & (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} & \twoheadrightarrow & (\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} & \twoheadrightarrow & \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K
\end{array}$$

of canonical morphisms compatibly extending those in (3.4.4)*, in which all squares are Cartesian, the vertical arrows in the upper-half are isomorphisms, the vertical arrows in the bottom-half are formal completions along locally closed subschemes, the horizontal arrows at the left-hand sides are open immersions with schematically dense images, the horizontal arrows at the right-hand sides are projective and surjective, and the compositions of horizontal arrows in the same rows are also open immersions with schematically dense images. This commutative diagram* (3.4.5) *is compatible with the commutative diagrams* (3.2.20) *and* (3.4.4) *along their common arrows.*

(4) *Let S be an irreducible noetherian normal scheme over* $\mathrm{Spec}(\mathcal{O}_K)$*, with generic point η, which is equipped with a morphism*

(3.4.6)
$$\eta \to \mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} K.$$

Let $(A_\eta, \lambda_\eta, i_\eta, \alpha_{\mathcal{H},\eta})$ denote the pullback of the tautological object of $\mathsf{M}_{\mathcal{H}}$ to η under (3.4.6)*. Suppose that, for each $\mathrm{j} \in \mathrm{J}$, we have a degenerating family $(G^\dagger_{\mathrm{j}}, \lambda^\dagger_{\mathrm{j}}, i^\dagger_{\mathrm{j}}, \alpha^\dagger_{\mathcal{H}_{\mathrm{j}}})$ of type $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$ over S, whose pullback $(G_{\mathrm{j},\eta}, \lambda_{\mathrm{j},\eta}, i_{\mathrm{j},\eta}, \alpha_{\mathcal{H}_{\mathrm{j}},\eta})$ to η defines a morphism*

(3.4.7)
$$\eta \to \mathsf{M}_{\mathcal{H}_{\mathrm{j}}} \underset{F_0}{\otimes} K$$

by the universal property of $\mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$, which we assume to coincide with the composition of (3.4.6) *with the canonical isomorphism $\mathsf{M}_{\mathcal{H}} \cong \mathsf{M}_{\mathcal{H}_{\mathrm{j}}}$ given by* [13, (2.1)]*. Suppose moreover that there exists an $\mathscr{L}$-set $(\underline{\mathscr{H}}^\dagger, \underline{\mathscr{F}}^\dagger, \underline{j}^\dagger)$ of polarized $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_S$-modules extending the pullback $(\underline{\mathscr{H}}_\eta, \underline{\mathscr{F}}_\eta, \underline{j}_\eta)$ of the $(\underline{\mathscr{H}}, \underline{\mathscr{F}}, \underline{j})$ over $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ (see Definition* 2.4.1 *and Lemma* 2.2.7*) and inducing compatible isomorphisms $\mathscr{F}^\dagger_{\Lambda_{\mathrm{j}}} \cong \underline{\mathrm{Lie}}^\vee_{G^{\dagger,\vee}_{\mathrm{j}}/S}$ and $\mathscr{H}^\dagger_{\Lambda_{\mathrm{j}}}/\mathscr{F}^\dagger_{\Lambda_{\mathrm{j}}} \cong \underline{\mathrm{Lie}}_{G^\dagger_{\mathrm{j}}/S}$ extending the canonical isomorphisms $\mathscr{F}_{\Lambda_{\mathrm{j}},\eta} \cong \underline{\mathrm{Lie}}^\vee_{G^\vee_{\mathrm{j},\eta}/\eta}$ and $\mathscr{H}_{\Lambda_{\mathrm{j}},\eta}/\mathscr{F}_{\Lambda_{\mathrm{j}},\eta} \cong \underline{\mathrm{Lie}}_{G_{\mathrm{j},\eta}/\eta}$, respectively, where Λ_{j} is as in Choices* 2.2.9*, for all $\mathrm{j} \in \mathrm{J}$; and that there exists a splitting structure $\{(\underline{\mathscr{F}}^{\dagger,i}_{[\tau]}, \underline{j}^{\dagger,i}_{[\tau]})\}_{[\tau]\in\Upsilon/\sim, 0\leq i<d_{[\tau]}}$ for $(\underline{\mathscr{H}}^\dagger, \underline{\mathscr{F}}^\dagger, \underline{j}^\dagger)$.*

Then (3.4.6) (*necessarily uniquely*) *extends to a morphism*

(3.4.8)
$$S \to \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$$

(over $\mathrm{Spec}(\mathcal{O}_K)$*), under which the above two tuples,* $(\underline{\mathscr{H}}^\dagger, \underline{\mathscr{F}}^\dagger, \underline{j}^\dagger)$ *and* $\{(\underline{\mathscr{F}}^{\dagger,i}_{[\tau]}, \underline{j}^{\dagger,i}_{[\tau]})\}_{[\tau]\in\Upsilon/\sim, 0\leq i<d_{[\tau]}}$*), are isomorphic to the pullbacks of the tautological tuples* $(\underline{\mathscr{H}}^{\mathrm{ext}}, \underline{\mathscr{F}}^{\mathrm{ext}}, \underline{j}^{\mathrm{ext}})$ *and* $\{(\underline{\mathscr{F}}^{\mathrm{ext},i}_{[\tau]}, \underline{j}^{\mathrm{ext},i}_{[\tau]})\}_{[\tau]\in\Upsilon/\sim, 0\leq i<d_{[\tau]}})$ *over* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$*, respectively, if and only if the following condition is satisfied at each geometric point* $\bar{s}$ *of* S*:*

Consider any dominant morphism $\mathrm{Spec}(V) \to S$ *centered at* $\bar{s}$*, where* V *is a complete discrete valuation ring with fraction field* $\widetilde{K}$*, algebraically closed residue field* k*, and discrete valuation* υ*. By the semistable reduction theorem (see, for example,* [5, Ch. I, Thm. 2.6] *or* [12, Thm. 3.3.2.4]*), up to replacing* $\widetilde{K}$ *with a finite extension field and replacing* V *accordingly, we may assume that the pullback of* A_η *to* $\mathrm{Spec}(\widetilde{K})$ *extends to a semi-abelian scheme* $G^\ddagger$ *over* $\mathrm{Spec}(V)$*. By the theory of Néron models (see* [3]*; cf.* [20, IX, 1.4], [5, Ch. I, Prop. 2.7], *or* [12, Prop. 3.3.1.5]*), the pullback of* $(A_\eta, \lambda_\eta, i_\eta, \alpha_{\mathcal{H},\eta})$ *to* $\mathrm{Spec}(\widetilde{K})$ *extends to a degenerating family* $(G^\ddagger, \lambda^\ddagger, i^\ddagger, \alpha^\ddagger_{\mathcal{H}})$ *of type* $\mathsf{M}_{\mathcal{H}}$ *over* $\mathrm{Spec}(V)$*, where* $\alpha^\ddagger_{\mathcal{H}}$ *is defined only over* $\mathrm{Spec}(\widetilde{K})$*, which defines an object of* $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(V)$ *corresponding to a tuple*

$$(B^\ddagger, \lambda_{B^\ddagger}, i_{B^\ddagger}, \underline{X}^\ddagger, \underline{Y}^\ddagger, \phi^\ddagger, c^\ddagger, c^{\vee,\ddagger}, \tau^\ddagger, [\alpha^{\natural,\ddagger}_{\mathcal{H}}])$$

in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(V)$ *under* [12, Thm. 5.3.1.19]*. Then* $[\alpha^{\natural,\ddagger}_{\mathcal{H}}]$ *determines a fully symplectic-liftable admissible filtration* $\mathsf{Z}^\ddagger_{\mathcal{H}}$*. Moreover, the étale sheaves* $\underline{X}^\ddagger$ *and* $\underline{Y}^\ddagger$ *are necessarily constant, because the base ring* V *is strict local. Hence, it makes sense to say we also have a uniquely determined torus argument* $\Phi^\ddagger_{\mathcal{H}}$ *at level* $\mathcal{H}$ *for* $\mathsf{Z}^\ddagger_{\mathcal{H}}$*. On the other hand, we have objects* $\underline{\Phi}_{\mathcal{H}}(G^\ddagger)$*,* $\underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G^\ddagger)}$*, and* $\underline{B}(G^\ddagger)$ *(see* [12, Constr. 6.3.1.1]*), which define objects* $\Phi^\ddagger_{\mathcal{H}}$*,* $\mathbf{S}_{\Phi^\ddagger_{\mathcal{H}}}$*, and in particular* $B^\ddagger : \mathbf{S}_{\Phi^\ddagger_{\mathcal{H}}} \to \mathrm{Inv}(V)$ *over the special fiber. Then* $\upsilon \circ B^\ddagger : \mathbf{S}_{\Phi^\ddagger_{\mathcal{H}}} \to \mathbb{Z}$ *defines an element of* $\mathbf{S}^\vee_{\Phi^\ddagger_{\mathcal{H}}}$*, where* $\upsilon : \mathrm{Inv}(V) \to \mathbb{Z}$ *is the homomorphism induced by the discrete valuation of* V*.*

Then the condition is that, for each $\mathrm{Spec}(V) \to S$ *as above (centered at* $\bar{s}$*), and for some (and hence every) choice of* $\delta^\ddagger_{\mathcal{H}}$*, there is a cone* $\sigma^\ddagger$ *in the cone decomposition* $\Sigma_{\Phi^\ddagger_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi^\ddagger_{\mathcal{H}}}$ *such that* $\overline{\sigma}^\ddagger$ *contains all* $\upsilon \circ B^\ddagger$ *obtained in this way. (As explained in the proof of* [12, Prop. 6.3.3.11]*, we may assume that* $\sigma^\ddagger$ *is minimal among such choices; also, it follows from the positivity of* $\tau^\ddagger$ *that* $\sigma^\ddagger \subset \mathbf{P}^+_{\Phi^\ddagger_{\mathcal{H}}}$*. Then the extended morphism* (3.4.8) *maps* $\bar{s}$ *to a geometric point over* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi^\ddagger_{\mathcal{H}}, \delta^\ddagger_{\mathcal{H}}, \sigma^\ddagger)]}$*; conversely, this property also characterizes the stratum* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi^\ddagger_{\mathcal{H}}, \delta^\ddagger_{\mathcal{H}}, \sigma^\ddagger)]}$ *of* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$*.)*

In particular, since this condition involves only $\mathcal{H}$*,* Σ*, and the linear algebraic data in Section* 2.1 *(such as* $\mathscr{L}$*) and Choices* 2.3.1*, the scheme* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ *depends (up to canonical isomorphism) only on these, but not on any auxiliary choices made in* [13, Sec. 7] *or any compatible collection* pol *of polarization function as in* [15, Sec. 2]*.*

Proof. By its very construction in Definition 3.1.3, we know $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ as a normal scheme flat over $\mathrm{Spec}(\mathcal{O}_K)$ and projective over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, with the tautological structures as described in the beginning of this theorem, which satisfies assertion (1).

The assertions (2) and (3) then follow from [13, Prop. 7.4, 8.1, 8.4, 8.7, 8.14, and 8.20; Thm. 9.13, 10.13, and 11.2; and Cor. 10.16, 10.18, and 11.9], [15, Thm. 6.1 (3) and (4)], the constructions summarized in Remark 3.2.19, and Theorem 3.3.1.

It remains to justify assertion (4). By [13, Thm. 11.4] and [15, Thm. 6.1(6)], the condition there is necessary and sufficient for (3.4.6) to extend to a morphism

$$S \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K. \tag{3.4.9}$$

By Proposition 3.1.2, the tautological tuple $(\underline{\mathscr{H}}^{\mathrm{ext}}, \underline{\mathscr{F}}^{\mathrm{ext}}, \underline{j}^{\mathrm{ext}})$ over $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ canonically descends to $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K$, whose pullback under (3.4.9) must be isomorphic to the $(\underline{\mathscr{H}}^{\dagger}, \underline{\mathscr{F}}^{\dagger}, \underline{j}^{\dagger})$ over S, by the density of η in S, and by the assumption that $(\underline{\mathscr{H}}^{\mathrm{ext}}, \underline{\mathscr{F}}^{\mathrm{ext}}, \underline{j}^{\mathrm{ext}})$ induces compatible isomorphisms $\mathscr{F}^{\dagger}_{\Lambda_{\mathrm{j}}} \cong \underline{\mathrm{Lie}}^{\vee}_{G^{\dagger,\vee}_{\mathrm{j}}/S}$ and $\mathscr{H}^{\dagger}_{\Lambda_{\mathrm{j}}}/\mathscr{F}^{\dagger}_{\Lambda_{\mathrm{j}}} \cong \underline{\mathrm{Lie}}_{G^{\dagger}_{\mathrm{j}}/S}$ extending the canonical isomorphisms $\mathscr{F}_{\Lambda_{\mathrm{j}},\eta} \cong \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}_{\mathrm{j},\eta}/\eta}$ and $\mathscr{H}_{\Lambda_{\mathrm{j}},\eta}/\mathscr{F}_{\Lambda_{\mathrm{j}},\eta} \cong \underline{\mathrm{Lie}}_{G_{\mathrm{j},\eta}/\eta}$, respectively, for all $\mathrm{j} \in \mathrm{J}$. Thus, the morphism (3.4.9) lifts to a morphism $S \to \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ as in (3.4.8) by the universal property of $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} = \mathrm{Spl}^{+}_{(\underline{\mathscr{H}}^{\mathrm{ext}}, \underline{\mathscr{F}}^{\mathrm{ext}}, \underline{j}^{\mathrm{ext}})/(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \otimes_{\mathcal{O}_{F_0,(p)}} \mathcal{O}_K)}$ (see (3.1.4)), as desired. □

Proposition 3.4.10 (cf. [13, Prop. 13.7, 13.9, and 13.15] and [15, Prop. 7.3 and 7.5])**.** *With the same setting as in Proposition* 2.4.17*, suppose moreover that* Σ *and* Σ' *are compatible collections of projective admissible rational polyhedral cone decomposition data for* $\mathsf{M}_{\mathcal{H}}$ *and* $\mathsf{M}_{\mathcal{H}'}$*, respectively, as in* [15, Def. 2.1 and 2.7]*, such that* Σ *is a g-refinement of* Σ' *as in* [12, Def. 6.4.3.3]*. Then the morphism* (2.4.18) *extends to a canonical projective morphism*

$$[\vec{g}]^{\mathrm{tor}} : \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma'}, \tag{3.4.11}$$

whose pullback from $\mathcal{O}_{F_0,(p)}$ *to* $\mathcal{O}_K$ *lifts to a canonical projective morphism*

$$[\vec{g}]^{\mathrm{spl,tor}} : \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H}',\Sigma'} \tag{3.4.12}$$

extending the morphism (2.4.19)*. The morphism* (3.4.11) (*resp.* (3.4.12)) *maps the* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$*-stratum* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ (*resp.* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$) *of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (*resp.* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$) *to the* $[(\Phi'_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma')]$*-stratum* $\vec{\mathsf{Z}}_{[(\Phi'_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma')]}$ (*resp.* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi'_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma')]}$) *of* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H}',\Sigma'}$ (*resp.* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H}',\Sigma'}$) *if and only if there are* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *and* $(\Phi'_{\mathcal{H}'}, \delta'_{\mathcal{H}'}, \sigma')$ *representing* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ *and* $[(\Phi'_{\mathcal{H}'}, \delta'_{\mathcal{H}'}, \sigma')]$*, respectively, such that* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *is a g-refinement of* $(\Phi'_{\mathcal{H}'}, \delta'_{\mathcal{H}'}, \sigma')$ *as in* [12, Def. 6.4.3.1]*. Also, the analogues of* [15, Prop. 7.5] *for* (3.4.11) *and* (3.4.12) *are true.*

Proof. The existence of the canonical morphisms (3.4.11) and (3.4.12) (with the desired properties) follows from Proposition 2.4.17 and its proof, and from comparisons of the universal properties of objects involved, using [13, Thm. 11.4] and [15, Thm. 6.1(6)], and using (4) of Theorem 3.4.1. As for the last statement, it follows from the same argument as in the proof of [15, Prop. 7.5], by showing that the formal completions of the toroidal compactifications along the pullbacks of strata of the corresponding minimal compactifications have the desired forms, using [13, Thm. 7.14 and 11.4], [15, Thm. 6.1], and Theorem 3.4.1. □

By the same arguments as in the proofs of [13, Prop. 14.1 and 14.2], using the fact that the squares in the commutative diagrams (3.2.20) and (3.4.5) are all Cartesian, we obtain the following two propositions:

Proposition 3.4.13 (cf. [13, Prop. 14.1])**.** *Suppose* Σ *is smooth as in* [12, Def. 6.3.3.4]. *Then* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ *is regular if and only if* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ *is.*

Proposition 3.4.14 (cf. [13, Prop. 14.2])**.** *Let* P *be the property of being one of the following: reduced, geometrically reduced, normal, geometrically normal, Cohen–Macaulay,* (R_0)*, geometric* (R_0)*,* (R_1)*, geometric* (R_1)*, and* (S_i)*, one property for each* $i \geq 0$ (*see* [7, IV-2, 5.7.2 and 5.8.2]). *Then the fiber of* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \mathrm{Spec}(\mathcal{O}_K)$ *over some point* s *of* $\mathrm{Spec}(\mathcal{O}_K)$ *satisfies property* P *if and only if the corresponding fiber of the open subscheme* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \mathrm{Spec}(\mathcal{O}_K)$ *over* s *does. If* Σ *is smooth as in* [12, Def. 6.3.3.4]*, then* P *can also be the property of being one of the following: regular, geometrically regular,* (R_i)*, and geometrically* (R_i)*, one property for each* $i \geq 0$.

Corollary 3.4.15 (cf. [13, Cor. 14.4])**.** *Suppose that the geometric fibers of* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \mathrm{Spec}(\mathcal{O}_K)$ *are reduced* (*resp. have integral local rings*). *Then all geometric fibers of* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \mathrm{Spec}(\mathcal{O}_K)$ *have the same number of connected* (*resp. irreducible*) *components, and the same is true for* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \mathrm{Spec}(\mathcal{O}_K)$.

Proof. By Proposition 3.4.14, the proper flat morphism $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \mathrm{Spec}(\mathcal{O}_K)$ has geometric fibers with reduced (resp. integral) local rings. So, by [6, Prop. 8.5.16], in its Stein factorization $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\mathrm{st}} \to \mathrm{Spec}(\mathcal{O}_K)$ (see [7, III-1, 4.3.3 and 4.3.4]), the second morphism is étale, while the first has connected and hence reduced (resp. integral) geometric fibers. Thus, the assertions for $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ follow.

The assertion for $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ concerning irreducible components then follows from the fiberwise density of $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ in $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ over $\mathrm{Spec}(\mathcal{O}_K)$ (see (2) of Theorem 3.4.1).

The assertion for $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ concerning connected components does not follow as easily, because an open dense subset of a connected set is not necessarily connected. Nevertheless, we have the following subtler argument: By (2) and (3) of Theorem 3.4.1, and by Artin's approximation (see [1, Thm. 1.12 and the proof of the corollaries in Sec. 2]), for each geometric point $\bar{s} \to \mathrm{Spec}(\mathcal{O}_K)$, and for each $x \in (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})_{\bar{s}}$, there exist an étale neighborhood $x \to U \to (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})_{\bar{s}}$ and an étale morphism $U \to (\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma))_{\bar{s}}$ (see Proposition 3.2.11), for some $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$, such that (by also approximating closed subschemes defining the boundary) the (open) preimages of $(\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}})_{\bar{s}}$ and $(\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_{\bar{s}}$ in U coincide with each other, and such that (up to replacing U with an open neighborhood of x) these étale morphisms have connected geometric fibers. Since $\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \hookrightarrow \vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ is fiberwise dense between schemes with geometrically irreducible fibers over $\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, since the formation of closures commutes with any flat base change (see [7, IV-2, 2.3.10]), and since x is arbitrary, the connected components of $(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})_{\bar{s}}$ are exactly the closures of those of $(\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}})_{\bar{s}}$. Since $\bar{s}$ is also arbitrary, the desired assertion still follows. □

4. Minimal compactifications

4.1. **Variants of Hodge invertible sheaves.** Unless otherwise specified, all tensor products of quasi-coherent sheaves in this section will be over their respective base schemes.

Definition 4.1.1. *The invertible sheaf* $\omega_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$ *over* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ *is the pullback of the ample invertible sheaf* $\omega_{\vec{\mathsf{M}}_{\mathcal{H}},\mathrm{J}}$ *over* $\vec{\mathsf{M}}_{\mathcal{H}}$ (*see* [13, Prop. 6.1]) *under the canonical morphism* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \vec{\mathsf{M}}_{\mathcal{H}}$*. Similarly, the invertible sheaf* $\omega_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ *over* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ *is the pullback of the invertible sheaf* $\omega_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ *over* $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (*cf.* [13, Prop. 7.11] *and* [15, Thm. 6.1(2)]) *under the canonical morphism* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$.

Remark 4.1.2. Since $\omega_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ and $\omega_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ are (by definition) the pullbacks of the ample invertible sheaf $\omega_{\vec{\mathsf{M}}^{\min}_{\mathcal{H}},\mathrm{J}}$ over $\vec{\mathsf{M}}^{\min}_{\mathcal{H}}$ (see [13, Prop. 6.4]), they are semiample in the sense that both $\omega^{\otimes N}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ and $\omega^{\otimes N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ are generated by their global sections (over their respective base schemes) for all sufficiently large N.

Definition 4.1.3. *Consider the invertible sheaf*

$$\omega^{\underline{\mu}}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}} := \omega^{\underline{\mu},+}_{(\underline{\mathscr{H}},\underline{\mathscr{F}},\underline{j})/(\vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K)} \tag{4.1.4}$$

over $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$*, which extends to the invertible sheaf*

$$\omega^{\underline{\mu}}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}} := \omega^{\underline{\mu},+}_{(\underline{\mathscr{H}}^{\mathrm{ext}},\underline{\mathscr{F}}^{\mathrm{ext}},\underline{j}^{\mathrm{ext}})/(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K)} \tag{4.1.5}$$

over $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$*, defined as in* (2.3.8) *and Proposition* 2.3.10 (*see also Definitions* 2.4.5 *and* 3.1.3) *for each positive* $\underline{\mu}$*. For each integer* k*, consider the invertible sheaf*

$$\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}},\mathrm{J}} := \omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}},\mathrm{J}} \otimes \omega^{\underline{\mu}}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}} \tag{4.1.6}$$

over $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$*, which extends to the invertible sheaf*

$$\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}} := \omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}} \otimes \omega^{\underline{\mu}}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}} \tag{4.1.7}$$

over $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$*. For simplicity, for each integer* N*, we shall abusively denote the* N*-th tensor powers of* (4.1.6) *and* (4.1.7) *by* $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$ *and* $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$*, respectively.*

Lemma 4.1.8. *For each positive* $\underline{\mu}$*, there exists some constant* $k_0(\underline{\mu}) \geq 0$ *such that* $\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$ *is ample for all* $k \geq k_0(\underline{\mu})$*. Consequently, there also exists some constant* $N_0(\underline{\mu}) \geq 0$ *such that* $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$ *is very ample for all* $k \geq k_0(\underline{\mu})$ *and* $N \geq N_0(\underline{\mu})$.

Proof. This is because $\omega^{\underline{\mu}}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}}$ is relatively ample over $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ (see Proposition 2.3.10), and because $\omega_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$ is (by definition) the pullback of the ample invertible sheaf $\omega_{\vec{\mathsf{M}}_{\mathcal{H}},\mathrm{J}}$ over $\vec{\mathsf{M}}_{\mathcal{H}}$ (see Definition 4.1.1 and [13, Prop. 6.1]). ☐

Definition 4.1.9. *For each cusp label $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$, and for integers k and N, we define as in Definition 4.1.3 the invertible sheaves $\omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$, $\omega^{\underline{\mu}}_{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}}$, $\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$, and $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$ over $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}} = \mathrm{Spl}^{+}_{(\,^{\sharp}\underline{\mathscr{H}},\,^{\sharp}\underline{\mathscr{F}},\,^{\sharp}\underline{j})/(\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K)}$ (see (3.2.2)).*

For each triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ such that its equivalence class $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ defines a stratum $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ of $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$, consider the structural morphisms

$$\vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}} \tag{4.1.10}$$

and

$$\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}} \tag{4.1.11}$$

(see Definition 3.2.3 and Remark 3.2.19), which are compatible with the structural morphism $\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$.

Lemma 4.1.12. *For each $\Lambda \in \mathscr{L}$, each $[\tau] \in \Upsilon/\sim$, and each integer i satisfying $0 \leq i < d_{[\tau]}$, consider the invertible sheaf*

$$\omega^{i}_{\Lambda,[\tau],\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}} := \wedge^{\mathrm{top}}\,(\mathscr{F}^{\mathrm{ext},i}_{\Lambda,[\tau]}/\mathscr{F}^{\mathrm{ext},i+1}_{\Lambda,[\tau]}) \tag{4.1.13}$$

over $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$, where

$$\{(\underline{\mathscr{F}}^{\mathrm{ext},i}_{[\tau]}, \underline{j}^{\mathrm{ext},i}_{[\tau]})\}_{[\tau]\in\Upsilon/\sim,0\leq i<d_{[\tau]}}$$

is the tautological splitting structure over $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ (see Theorem 3.4.1); and consider the invertible sheaf

$$\omega^{i}_{\Lambda,[\tau],\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}} := \wedge^{\mathrm{top}}\,(\,^{\sharp}\mathscr{F}^{i}_{\Lambda,[\tau]}/\,^{\sharp}\mathscr{F}^{i+1}_{\Lambda,[\tau]}) \tag{4.1.14}$$

over $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$, where

$$\{(\,^{\sharp}\underline{\mathscr{F}}^{i}_{[\tau]},\,^{\sharp}\underline{j}^{i}_{[\tau]})\}_{[\tau]\in\Upsilon/\sim,0\leq i<d_{[\tau]}}$$

is the tautological splitting structure over $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}} = \mathrm{Spl}^{+}_{(\,^{\sharp}\underline{\mathscr{H}},\,^{\sharp}\underline{\mathscr{F}},\,^{\sharp}\underline{j})/(\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K)}$ (see Definition 3.2.1). Then the pullback of $\omega^{i}_{\Lambda,[\tau],\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}}$ under the canonical morphism $\vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ induced by the inverse of (3.3.2) is isomorphic to the pullback of $\omega^{i}_{\Lambda,[\tau],\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}}$ under the morphism (4.1.10).

Proof. Consider the pullback $^{\natural}\mathscr{F}^{i}_{\Lambda,[\tau]}$ of $\mathscr{F}^{\mathrm{ext},i}_{\Lambda,[\tau]}$ to $\mathfrak{X}^{\mathrm{spl}} = (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$, as in Section 3.3. By assigning $^{\sharp}\mathscr{F}^{i}_{\Lambda,[\tau]}$ and $^{\flat}\mathscr{F}^{i}_{\Lambda,[\tau]}$ to $^{\natural}\mathscr{F}^{i}_{\Lambda,[\tau]}$ as in (3.3.12) and (3.3.13), we have a short exact sequence $0 \to {}^{\sharp}\mathscr{F}^{i}_{\Lambda,[\tau]} \to {}^{\natural}\mathscr{F}^{i}_{\Lambda,[\tau]} \to {}^{\flat}\mathscr{F}^{i}_{\Lambda,[\tau]} \to 0$ of locally free $\mathscr{O}_{\mathfrak{X}^{\mathrm{spl}}}$-modules as in (3.3.20), which induces an isomorphism $\wedge^{\mathrm{top}}\,(\,^{\natural}\mathscr{F}^{i}_{\Lambda,[\tau]}) \cong \wedge^{\mathrm{top}}\,(\,^{\sharp}\mathscr{F}^{i}_{\Lambda,[\tau]}) \otimes \wedge^{\mathrm{top}}\,(\,^{\flat}\mathscr{F}^{i}_{\Lambda,[\tau]})$ of invertible sheaves over $\mathfrak{X}^{\mathrm{spl}}$. By Corollary 3.3.16 and its proof, $\wedge^{\mathrm{top}}\,(\,^{\flat}\mathscr{F}^{i}_{\Lambda,[\tau]}) \cong \wedge^{\mathrm{top}}\,(Y_{\Lambda,[\tau]} \underset{\mathbb{Z}_p}{\otimes} \mathscr{O}_{\mathfrak{X}^{\mathrm{spl}}}) \cong (\wedge^{\mathrm{top}}_{\mathbb{Z}_p}\,(Y_{\Lambda,[\tau]})) \underset{\mathbb{Z}_p}{\otimes} \mathscr{O}_{\mathfrak{X}^{\mathrm{spl}}}$ is trivial. Then it suffices to note that, by the construction of (3.3.2) (see the proof of Theorem 3.3.1), the $^{\sharp}\mathscr{F}^{i}_{\Lambda,[\tau]}$ over $\mathfrak{X}^{\mathrm{spl}}$ is canonically isomorphic to the pullback of the $^{\sharp}\mathscr{F}^{i}_{\Lambda,[\tau]}$ over $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$ under the composition of (3.3.2) and (4.1.10). ☐

Corollary 4.1.15. *For each $\underline{\mu}$, and for any integers k and N, the pullback of the invertible sheaf $\omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ (resp. $\omega^{\underline{\mu}}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}}$, resp. $\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$, resp. $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$) to $\vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ via the canonical morphism $\vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ induced by the inverse of* (3.3.2) *is isomorphic to the pullback of $\omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$ (resp. $\omega^{\underline{\mu}}_{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}}$, resp. $\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$, resp. $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$) under the morphism* (4.1.10).

Proof. The case for $\omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ follows from [12, Lem. 7.1.2.1, and the proof of Thm. 7.2.4.1], and from the definitions (see Definitions 4.1.1 and 4.1.9). The case for $\omega^{\underline{\mu}}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}}$ follows from Lemma 4.1.12, and from the definition of $\omega^{\underline{\mu}}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}}$ (see (2.3.8)). The remaining cases then follow from these two cases, by definition. □

Corollary 4.1.16. *For each positive $\underline{\mu}$, and for each cusp label $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$, there exists some constant $k_{(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})}(\underline{\mu}) \geq 0$ such that, for each triple $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ defining a stratum $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ of $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ above the stratum $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of $\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}$ (see Definition* 3.1.8, [13, Thm. 12.1], *and* [15, Thm. 6.1]), *the pullback of $\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ to $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is semiample for all $k \geq k_{(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})}(\underline{\mu})$, and is isomorphic to the pullback of an ample invertible sheaf $\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$ under the structural morphism* (4.1.11). *Consequently, there also exists some constant $N_{(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})}(\underline{\mu}) \geq 0$ such that the pullback of $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ to $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is generated by its global sections for all $k \geq k_{(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})}(\underline{\mu})$ and $N \geq N_{(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})}(\underline{\mu})$ (see Remark* 4.1.2), *and is isomorphic to the pullback of a very ample invertible sheaf $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$ under the structural morphism* (4.1.11)

Proof. This follows from Corollary 4.1.15, and from the same argument as in the proof of Lemma 4.1.8. □

Lemma 4.1.17. *For each $\Lambda \in \mathscr{L}$, each $[\tau] \in \Upsilon/\sim$, and each integer i satisfying $0 \leq i < d_{[\tau]}$, the pullback $\omega^{i}_{\Lambda,[\tau],\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}\otimes_{F_0} K}$ of $\omega^{i}_{\Lambda,[\tau],\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}}$ (see* (4.1.13)) *to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}\otimes_{F_0} K$ (see* (3.4.2)) *descends to an invertible sheaf $\omega^{i}_{\Lambda,[\tau],\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}\otimes_{F_0} K}$ over $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}\otimes_{F_0} K$.*

Proof. As in the proof of [12, Thm. 7.2.4.1], it suffices to note that the pullback of $\omega^{i}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\Lambda,[\tau]}$ to $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}\otimes_{F_0} K$ descends to $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}\otimes_{F_0} K$, by Lemma 4.1.12. □

Corollary 4.1.18. *For each $\underline{\mu}$, and for all k and N, the pullback $\omega^{\otimes(k,\underline{\mu})N}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}\otimes_{F_0} K,\mathrm{J}}$ of $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}\otimes_{F_0} K$ (see* (3.4.2)) *descends to an invertible sheaf $\omega^{\otimes(k,\underline{\mu})N}_{\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}\otimes_{F_0} K,\mathrm{J}}$ over $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}\otimes_{F_0} K$. Consequently, for all sufficiently large integer k (depending on $\underline{\mu}$), the invertible sheaf $\omega^{\otimes(k,\underline{\mu})N}_{\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}\otimes_{F_0} K,\mathrm{J}}$ over $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}\otimes_{F_0} K$ is ample, and so that its pullback $\omega^{\otimes(k,\underline{\mu})N}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}\otimes_{F_0} K,\mathrm{J}}$ to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}\otimes_{F_0} K$ is semiample.*

Proof. By Definitions 4.1.1 and 4.1.3, and by the definition of $\omega^{\underline{\mu}}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}}$ (see (2.3.8)), this follows from Lemma 4.1.17. □

Corollary 4.1.19. *For each positive $\underline{\mu}$, there exist integers $k_1(\underline{\mu}) \geq k_0(\underline{\mu})$ and $N_1(\underline{\mu}) \geq N_0(\underline{\mu})$ such that, for all integers $k \geq k_1(\underline{\mu})$ and $N \geq N_1(\underline{\mu})$, the pullback of the invertible sheaf $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ to $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \cup (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \cong \vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \cup (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{F_0}{\otimes} K)$ (glued over their common open subscheme $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong \mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} K$) is generated by its global sections and descends to a very ample invertible sheaf over $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \cup (\mathsf{M}^{\mathrm{min}}_{\mathcal{H}} \underset{F_0}{\otimes} K)$.*

Proof. This follows from Corollary 4.1.18 and the same argument as in the proof of Lemma 4.1.8. □

Remark 4.1.20. The constants in Lemma 4.1.8 and in Corollaries 4.1.16 and 4.1.19 depend on the integral PEL datum $(\mathcal{O}, \star, \langle \cdot, \cdot \rangle, h_0)$, on the choices of J and $\mathscr{L}_{\mathrm{J}}$ (see Choices 2.2.9), on the level $\mathcal{H}$ (see Choices 2.2.10), on the choices of the integers $\{a_{\mathrm{j}}\}_{\mathrm{j}\in\mathrm{J}}$ as in [13, Lem. 5.30], on the choices of K and the ordering of $\tau_{[\tau],0}, \tau_{[\tau],1}, \ldots, \tau_{[\tau],d_{[\tau]}-1}$ in $[\tau]$ for all $[\tau] \in \Upsilon/\sim$ (see Choices 2.3.1), and on $\underline{\mu}$.

Lemma 4.1.21. *The canonical restriction map*

$$\Gamma(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}) \to \Gamma(\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \cup (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}), \omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}})$$

is bijective for all $\underline{\mu}$, k, and N.

Proof. Since $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ is noetherian and normal by construction, which is (S_2) at all points of codimension at least two by Serre's criterion (see [7, IV-2, 5.8.6]), and since the complement of $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \cup (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ in $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ has codimension at least two (because $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ is fiberwise dense in $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$, by Theorem 3.4.1), this follows from [9, Prop. 1.11 and Thm. 3.8]. □

Proposition 4.1.22. *For each positive $\underline{\mu}$, given any integers $k \geq k_1(\underline{\mu})$ and $N \geq N_1(\underline{\mu})$, the canonical morphism*

$$\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \cup (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \to \mathbb{P}_{\mathrm{Spec}(\mathcal{O}_K)}(\Gamma(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}})) \tag{4.1.23}$$

induces a canonical open immersion

$$\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \cup (\mathsf{M}^{\mathrm{min}}_{\mathcal{H}} \underset{F_0}{\otimes} K) \hookrightarrow \mathbb{P}_{\mathrm{Spec}(\mathcal{O}_K)}(\Gamma(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}})) \tag{4.1.24}$$

whose pre-composition with the canonical morphism

$$\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \cup (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \to \vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \cup (\mathsf{M}^{\mathrm{min}}_{\mathcal{H}} \underset{F_0}{\otimes} K) \tag{4.1.25}$$

is (4.1.23)*. Let us define $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$ to be the normalization of the closure of the image of* (4.1.24) *in $\mathbb{P}_{\mathrm{Spec}(\mathcal{O}_K)}(\Gamma(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}))$. Then we have a canonical open immersion*

$$\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \cup (\mathsf{M}^{\mathrm{min}}_{\mathcal{H}} \underset{F_0}{\otimes} K) \hookrightarrow \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}, \tag{4.1.26}$$

with schematically dense image (by definition of $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$*), whose pre-composition with* (4.1.25) *defines a canonical morphism*

$$\vec{\oint}^{\mathrm{spl,pre}}_{\mathcal{H},(k,\underline{\mu})N} : \vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \cup (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}) \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N} \tag{4.1.27}$$

with schematically dense image. The pullback of $\mathcal{O}_{\mathbb{P}_{\mathrm{Spec}(\mathcal{O}_K)}(\Gamma(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}))}(1)$ *to* $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$*, which we abusively denote by* $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N},\mathrm{J}}$ *(before defining* $\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N},\mathrm{J}}$*), is an ample invertible sheaf, whose further pullback to* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \cup (\mathsf{M}^{\mathrm{min}}_{\mathcal{H}} \underset{F_0}{\otimes} K)$ *under* (4.1.26) *is the very ample invertible sheaf in Corollary* 4.1.19*.*

Proof. The existence of the canonical morphism (4.1.23) and the induced open immersion (4.1.24) follows from Lemma 4.1.8, Corollaries 4.1.18 and 4.1.19, and Lemma 4.1.21. The rest of the assertions are self-explanatory. □

Choices 4.1.28. From now on, for each positive $\underline{\mu}$, we shall fix the choices of some integers $k_2(\underline{\mu}) \geq k_1(\underline{\mu})$ and $N_2(\underline{\mu}) \geq N_1(\underline{\mu})$ such that $k_2(\underline{\mu}) \geq k_{(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})}(\underline{\mu})$ and $N_2(\underline{\mu}) \geq N_{(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})}(\underline{\mu})$ for all cusp labels $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ (see Corollary 4.1.16).

We will show in the next section that, when $k \geq k_2(\underline{\mu})$ and $N \geq N_2(\underline{\mu})$, the morphism (4.1.27) extends to a morphism $\vec{\oint}^{\mathrm{spl}}_{\mathcal{H}} : \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}$, whose target is (up to unique isomorphism) independent of the choices of k and N.

4.2. **Semiampleness and projective spectra.**

Throughout this section, we shall fix the choice of a positive $\underline{\mu}$, and assume that $k \geq k_2(\underline{\mu})$ and $N \geq N_2(\underline{\mu})$, where $k_2(\underline{\mu})$ and $N_2(\underline{\mu})$ are as in Choices 4.1.28.

Let $\mathrm{Graph}(\vec{\oint}^{\mathrm{spl,pre}}_{\mathcal{H},(k,\underline{\mu})N})$ denote the graph of the canonical morphism $\vec{\oint}^{\mathrm{spl,pre}}_{\mathcal{H},(k,\underline{\mu})N}$ in (4.1.27), which we view as a locally closed subscheme of $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \underset{\mathrm{Spec}(\mathcal{O}_K)}{\times} \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$, which is isomorphic to the open dense subscheme $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \cup (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ of $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ via the first projection, and has schematically dense image via the second projection (see Proposition 4.1.22). Let us denote by

$$\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N}$$

the normalization of the necessarily reduced schematic closure of $\mathrm{Graph}(\vec{\oint}^{\mathrm{spl,pre}}_{\mathcal{H},(k,\underline{\mu})N})$ in $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \underset{\mathrm{Spec}(\mathcal{O}_K)}{\times} \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$. Then the projections from $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \underset{\mathrm{Spec}(\mathcal{O}_K)}{\times} \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$ to its two factors induce canonical proper surjections

$$\vec{\oslash}^{\mathrm{spl}}_{\mathcal{H},(k,\underline{\mu})N} : \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N} \to \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \tag{4.2.1}$$

and

$$\vec{\oint}^{\mathrm{spl}}_{\mathcal{H},(k,\underline{\mu})N} : \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N} \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}. \tag{4.2.2}$$

Lemma 4.2.3. *The canonical morphisms*

$$\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}} \to (\vec{\oslash}^{\mathrm{spl}}_{\mathcal{H},(k,\underline{\mu})N})_* \mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N}} \tag{4.2.4}$$

and

$$\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}} \to (\vec{\oint}^{\mathrm{spl}}_{\mathcal{H},(k,\underline{\mu})N})_* \mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N}} \tag{4.2.5}$$

induced by (4.2.1) *and* (4.2.2), *respectively, are isomorphisms. Consequently, the morphisms* (4.2.1) *and* (4.2.2) *are their own Stein factorizations* (*see* [7, III-1, 4.3.3 and 4.3.4]), *by abuse of language, and their geometric fibers are all connected.*

Proof. Since (4.2.1) is proper, it induces a Stein factorization

$$\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N} \to \underline{\mathrm{Spec}}_{\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}}}\left((\vec{\partial}^{\mathrm{spl}}_{\mathcal{H},(k,\underline{\mu})N})_* \mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N}}\right) \to \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}, \tag{4.2.6}$$

and we need the second (finite) morphism to be an isomorphism. Since $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N}$ is normal, the second scheme in (4.2.6) is normal. Since (4.2.1) induces the identity morphism over the open dense subscheme $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$, the second morphism in (4.2.6) is an isomorphism by Zariski's main theorem (see [7, III-1, 4.4.3, 4.4.11]). This shows that (4.2.4) is an isomorphism. The argument for (4.2.5) is similar. □

For each (locally closed) stratum $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ of $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ as in (3.1.9), consider the locally closed subscheme

$$\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N} := (\vec{\partial}^{\mathrm{spl}}_{\mathcal{H},(k,\underline{\mu})N})^{-1}(\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}) \tag{4.2.7}$$

of $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N}$ with its canonical reduced subscheme structure. Then we have a disjoint union

$$\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N} \tag{4.2.8}$$

compatible with (3.1.9), [13, Thm. 9.13], and [15, Thm. 6.1].

For each stratum $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$, we have an induced proper morphism

$$(\vec{\partial}^{\mathrm{spl}}_{\mathcal{H},(k,\underline{\mu})N})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} : (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N}} \to (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \tag{4.2.9}$$

between the formal completions. By Lemma 4.2.3 and [7, III-1, 4.1.5], the isomorphism (4.2.4) induces a canonical isomorphism

$$\mathscr{O}_{(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}} \xrightarrow{\sim} \left((\vec{\partial}^{\mathrm{spl}}_{\mathcal{H},(k,\underline{\mu})N})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}\right)_* \mathscr{O}_{(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N}}}. \tag{4.2.10}$$

So, the pullback of any $f \in \Gamma(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}})$ to $(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N}}$ is determined by its pullback to $(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$, which defines the Fourier–Jacobi expansions of f as in [12, Sec. 7.1.2] and [13, Sec. 12]. By the same arguments there, we obtain the following:

Proposition 4.2.11 (cf. [12, Prop. 7.1.2.13] and [13, Prop. 12.10])**.** *The pullback of each* $f \in \Gamma(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}})$ *to the subscheme* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N}$ *of* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N}$ *is* **constant along the fibers** *of the structural morphism*

$$\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N} \xrightarrow{\substack{\text{restriction of}\\(4.2.1)}} \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \xrightarrow[\sim]{(3.3.3)^{-1}} \vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}. \tag{4.2.12}$$

Corollary 4.2.13. *The restriction of* (4.2.2) *to* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N}$ *induces a canonical morphism* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N} \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$*, which factors through a morphism* $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}} \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$*. Consequently, the stratum* $\mathsf{Z}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} := \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \underset{F_0}{\otimes} K$ *of* $\mathsf{M}^{\min}_{\mathcal{H}} \underset{F_0}{\otimes} K$ *is dense in the schematic image of* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N}$ *under* (4.2.2)*.*

Proof. The first assertion follows from Proposition 4.2.11. By [13, Thm. 12.1] and [15, Thm. 6.1 (3) and (5)], the restriction of (4.2.2) to the stratum

$$\mathsf{Z}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} := \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \underset{F_0}{\otimes} K \cong \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$$

of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{F_0}{\otimes} K \cong \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (see Definition 3.1.8 and (4.2.7)) induces a canonical surjection $\mathsf{Z}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \twoheadrightarrow \mathsf{Z}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. Hence, the last assertion follows from the flatness of $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$ over $\mathrm{Spec}(\mathcal{O}_K)$. □

For each stratum $\mathsf{Z}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of $\mathsf{M}^{\min}_{\mathcal{H}} \underset{F_0}{\otimes} K$ as in Corollary 4.2.13, consider its closure $\overline{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ in $\mathsf{M}^{\min}_{\mathcal{H}} \underset{F_0}{\otimes} K$ and its closure $\vec{\overline{\mathsf{Z}}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N}$ in $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$ under the open immersion (4.1.26) with schematically dense image. Then we define a locally closed subscheme

$$\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N} := \vec{\overline{\mathsf{Z}}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N} - \bigcup_{\mathsf{Z}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \not\subseteq \overline{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}} \vec{\overline{\mathsf{Z}}}^{\mathrm{spl}}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})],(k,\underline{\mu})N} \tag{4.2.14}$$

of $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$ (cf. [13, (6.8)]).

Proposition 4.2.15 (cf. [13, Thm. 12.1] and [15, Thm. 6.1 (3) and (5)])**.** *The locally closed subschemes* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N}$ *of* $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$ *form a stratification*

$$\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N}, \tag{4.2.16}$$

with incidence relations similar to those in [12, Thm. 7.2.4.1 (4) and (5)]*,* [13, Thm. 12.1]*, and* [15, Thm. 6.1(3)]*. For each representative* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ *such that* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ *labels a stratum* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *of* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ *as in* (3.1.9)*, the restriction of the canonical morphism* (4.2.2) *to the corresponding stratum* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N}$ *of* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N}$ *induces a canonical surjection*

$$\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N} \twoheadrightarrow \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N}, \tag{4.2.17}$$

which is proper when σ *is top-dimensional in* $\Sigma_{\Phi_{\mathcal{H}}}$*.*

Proof. By Corollary 4.2.13 and its proof, the restriction of (4.2.2) to $\mathsf{Z}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ induces a canonical surjection $\mathsf{Z}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \twoheadrightarrow \mathsf{Z}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$, and $\mathsf{Z}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ is dense in the schematic image of $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N}$ under (4.2.2). Since the morphism (4.2.2) is proper and surjective, and since the disjoint union (4.2.8) is the pullback of the stratification (3.1.9), it follows that $\mathsf{Z}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \cong \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$

as subschemes of $\mathsf{M}^{\min}_{\mathcal{H}} \underset{F_0}{\otimes} K \cong \vec{\mathsf{M}}^{\mathrm{spl},\min}_{\mathcal{H},(k,\underline{\mu})N} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, and that $\mathsf{Z}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ is dense in $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N}$. Hence, the union in (4.2.16) defines the desired stratification of $\vec{\mathsf{M}}^{\mathrm{spl},\min}_{\mathcal{H},(k,\underline{\mu})N}$. As for the properness of (4.2.17) when σ is top-dimensional in $\Sigma_{\Phi_{\mathcal{H}}}$, it follows from that of (4.2.2), because then $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N}$ is closed in the preimage $(\vec{\oint}^{\mathrm{spl}}_{\mathcal{H},(k,\underline{\mu})N})^{-1}(\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N})$ by the other assertions we have proved. □

Corollary 4.2.18 (cf. [12, Cor. 7.2.3.12] and [13, Cor. 12.12]). *The morphism* (4.2.17) *factors through* (4.2.12) *and defines a canonical surjection*

$$\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}} \twoheadrightarrow \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N}. \tag{4.2.19}$$

Under the running assumption that $k \geq k_2(\underline{\mu}) \geq k_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}(\underline{\mu})$ and $N \geq N_2(\underline{\mu}) \geq N_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}(\underline{\mu})$ (see Choices 4.1.28*), this surjection is finite and induces a canonical isomorphism from $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$ to the normalization of $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N}$.*

Proof. The first assertion follows from Corollary 4.2.13. Since the morphisms (4.2.12), for all possible σ, all factor through the same induced morphism (4.2.19) (by the same argument of relating every two cones by a sequences of inclusions of closures, as in the paragraph following [12, Rem. 7.1.2.5]), by taking σ to be top-dimensional in $\Sigma_{\Phi_{\mathcal{H}}}$, which necessarily satisfies $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$, it follows from Proposition 4.2.15 that the induced morphism (4.2.19) is proper. Since (by Corollary 4.1.15) the pullback of $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ to $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N}$ descends to $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$ (via (4.2.12)), the pullback of $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N},\mathrm{J}}$ (see Proposition 4.1.22) under (4.2.19) is isomorphic to the invertible sheaf $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$ over $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$, which is ample (by Corollary 4.1.16) under the assumption that $k \geq k_2(\underline{\mu}) \geq k_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}(\underline{\mu})$ and $N \geq N_2(\underline{\mu}) \geq N_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}(\underline{\mu})$. This shows that the proper morphism (4.2.19) is finite, by [7, II, 5.1.6, and III-1, 4.4.2]. Since (4.2.19) induces in characteristic zero the canonical isomorphism $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}} \underset{F_0}{\otimes} K \xrightarrow{\sim} \mathsf{Z}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} = \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \underset{F_0}{\otimes} K$ (see [12, Cor. 7.2.3.18]), the second assertion follows from Zariski's main theorem (see [7, III-1, 4.4.3, 4.4.11]), as desired. □

Proposition 4.2.20 (cf. [12, Prop. 7.2.3.16] and [13, Prop. 12.14]). *Let $\bar{x}$ be a geometric point of $\vec{\mathsf{M}}^{\mathrm{spl},\min}_{\mathcal{H},(k,\underline{\mu})N}$ over the $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$-stratum $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N}$. Let $(\vec{\mathsf{M}}^{\mathrm{spl},\min}_{\mathcal{H},(k,\underline{\mu})N})^{\wedge}_{\bar{x}}$ denote the completion of the strict localization of $\vec{\mathsf{M}}^{\mathrm{spl},\min}_{\mathcal{H},(k,\underline{\mu})N}$ at $\bar{x}$, let*

$$(\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N})^{\wedge}_{\bar{x}} := \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N} \underset{\vec{\mathsf{M}}^{\mathrm{spl},\min}_{\mathcal{H},(k,\underline{\mu})N}}{\times} (\vec{\mathsf{M}}^{\mathrm{spl},\min}_{\mathcal{H},(k,\underline{\mu})N})^{\wedge}_{\bar{x}},$$

and let

$$(\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}})^{\wedge}_{\bar{x}} := \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}} \underset{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N}}{\times} (\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N})^{\wedge}_{\bar{x}}.$$

For each $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}$, let $\vec{\Psi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ be as in (3.2.15)*, and let $(\underline{\vec{\mathrm{FJ}}}^{\mathrm{spl},(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$ denote the pullback of*

$$\underline{\vec{\mathrm{FJ}}}^{\mathrm{spl},(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} := (\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}})_*(\vec{\Psi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))$$

under the canonical morphism $(\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}})^{\wedge}_{\bar{x}} \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$*. Then we have a canonical isomorphism*

$$\mathscr{O}_{(\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N})^{\wedge}_{\bar{x}}} \cong \Big(\prod_{\ell \in \mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}} (\underline{\vec{\mathrm{FJ}}}^{\mathrm{spl},(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}} \Big)^{\Gamma_{\Phi_{\mathcal{H}}}}, \tag{4.2.21}$$

where $\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}} := \{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}} : \langle \ell, y\rangle \geq 0 \ \forall y \in \mathbf{P}_{\Phi_{\mathcal{H}}}\}$ *as usual, which is adic if we interpret the product on the right-hand side as the completion of the elements that are finite sums with respect to the ideal generated by the elements with zero constant terms (i.e., with zero projection to* $(\underline{\vec{\mathrm{FJ}}}^{\mathrm{spl},(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$*). Then the isomorphism* (4.2.21) *induces a homomorphism* $\big((\underline{\vec{\mathrm{FJ}}}^{\mathrm{spl},(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}\big)^{\Gamma_{\Phi_{\mathcal{H}}}} \to \mathscr{O}_{(\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N})^{\wedge}_{\bar{x}}}$*, whose source is canonically isomorphic to* $\mathscr{O}_{(\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}})^{\wedge}_{\bar{x}}}$ *(by Corollary* 4.2.18 *and Zariski's main theorem; see* [7, III-1, 4.4.3, 4.4.11]*). This homomorphism defines a* **structural morphism** $(\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N})^{\wedge}_{\bar{x}} \to (\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}})^{\wedge}_{\bar{x}}$*, whose pre-composition with the canonical morphism* $(\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N})^{\wedge}_{\bar{x}} \to (\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N})^{\wedge}_{\bar{x}}$ *defines a canonical morphism* $(\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N})^{\wedge}_{\bar{x}} \to (\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}})^{\wedge}_{\bar{x}}$*, which is then an isomorphism because its pre-composition with the formal completion* $(\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}})^{\wedge}_{\bar{x}} \to (\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N})^{\wedge}_{\bar{x}}$ *of* (4.2.19) *is the identity morphism on* $(\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}})^{\wedge}_{\bar{x}}$*. Consequently, this last completion of* (4.2.19) *is also an isomorphism.*

Proof. Using the canonical isomorphisms (4.2.10), the same argument as in the proof of [12, Prop. 7.2.3.16] works here. □

Remark 4.2.22. As remarked in the proof of [13, Prop. 12.14], we do not need to know a priori that (4.2.19) induces a bijection on geometric points. Also, by the same argument as in the proof of Corollary 4.2.18, we may remove the dependence on the second assertion of [13, Lem. 12.9] from the proof of [13, Prop. 12.14].

Corollary 4.2.23 (cf. [13, Thm. 12.16] and [15, Thm. 6.1(5)])**.** *In* (4.2.16)*, each stratum* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N}$ *is canonically isomorphic to* $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$*. The canonical surjection* (4.2.17) *can be identified with the composition of the canonical morphism* (4.2.12) *with the above-mentioned isomorphism* $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}} \xrightarrow{\sim} \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],(k,\underline{\mu})N}$.

Proof. As in the proof of [13, Thm. 12.16], it suffices to show that (4.2.19) is an isomorphism. Since this can be verified over formal completions of strict local rings, this follows from Proposition 4.2.20, as desired. □

Corollary 4.2.24. *With the same setting as in Proposition* 4.2.20*, let*

$$(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N})^{\wedge}_{\bar{x}} := \Big((\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)],(k,\underline{\mu})N}} \Big) \underset{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}}{\times} (\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}})^{\wedge}_{\bar{x}}$$

and

$$(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\bar{x}} := \Big((\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \Big) \underset{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}}{\times} (\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}})^{\wedge}_{\bar{x}}.$$

The canonical morphism

$$(\vec{\oint}^{\mathrm{spl}}_{\mathcal{H},(k,\underline{\mu})N})^{\wedge}_{\bar{x}} : (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N})^{\wedge}_{\bar{x}} \to (\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N})^{\wedge}_{\bar{x}}$$

induced by (4.2.2) *factors as the composition of the canonical morphism*

$$(\vec{\partial}^{\mathrm{spl}}_{\mathcal{H},(k,\underline{\mu})N})^{\wedge}_{\bar{x}} : (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N})^{\wedge}_{\bar{x}} \to (\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\bar{x}}$$

induced by (4.2.9) *with a canonical morphism*

$$(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma})^{\wedge}_{\bar{x}} \to (\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N})^{\wedge}_{\bar{x}}.$$

Proof. By treating all objects as formal schemes over $(\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}})^{\wedge}_{\bar{x}}$, this follows from the explicit description (4.2.21) of $\mathscr{O}_{(\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N})^{\wedge}_{\bar{x}}}$. □

Proposition 4.2.25. *The proper morphism* (4.2.1) *is an isomorphism, and hence the morphism* (4.2.2) *descends to a canonical morphism*

$$\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N} \tag{4.2.26}$$

extending (4.1.27), *under which* $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ (*see Definition* 4.1.3) *is isomorphic to the pullback of* $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N},\mathrm{J}}$ (*see Proposition* 4.1.22).

Proof. Since $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N}$ is by definition the normalization of the schematic closure of $\mathrm{Graph}(\vec{\oint}^{\mathrm{spl,pre}}_{\mathcal{H},(k,\underline{\mu})N})$ in $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \underset{\mathrm{Spec}(\mathcal{O}_K)}{\times} \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$, Corollary 4.2.24 shows that the proper morphism (4.2.1) is an isomorphism after pullback to an fpqc covering of $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$, which then forces (4.2.1) itself to be an isomorphism. □

Corollary 4.2.27. *The invertible sheaf* $\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ *over* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ *is semiample.*

Proof. Since the invertible sheaf $\omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N},\mathrm{J}}$ over $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$ is ample, this follows from Proposition 4.2.25. □

Corollary 4.2.28 (cf. [13, Cor. 12.5])**.** $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{F}_p$ *is dense in* $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N} \underset{\mathbb{Z}}{\otimes} \mathbb{F}_p$.

Proof. Since $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{F}_p$ is dense in $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma,(k,\underline{\mu})N} \underset{\mathbb{Z}}{\otimes} \mathbb{F}_p$ by (2) of Theorem 3.4.1, this follows from Proposition 4.2.15. □

Lemma 4.2.29. *For each* $\Lambda \in \mathscr{L}$, *each* $[\tau] \in \Upsilon/\sim$, *and each integer* i *satisfying* $0 \leq i < d_{[\tau]}$, *the invertible sheaf* $\omega^{i}_{\Lambda,[\tau],\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}}$ (*see* (4.1.13)) *descends to an invertible sheaf* $\omega^{i}_{\Lambda,[\tau],\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}}$ *over* $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$ *via the canonical morphism* (4.2.26).

Proof. By Lemma 4.1.17 and Corollary 4.2.28, and by the same argument as in the proof of [12, Thm. 7.2.4.1], it suffices to note that the pullback of each of these sheaves to each $\vec{\mathfrak{X}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ descends to $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$, by Lemma 4.1.12. □

Corollary 4.2.30. *For all positive* $\underline{\mu}'$ *and all integers* k' *and* N', *the invertible sheaf* $\omega^{\otimes k'}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ (*resp.* $\omega^{\underline{\mu}'}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}}$, *resp.* $\omega^{\otimes(k',\underline{\mu}')}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$, *resp.* $\omega^{\otimes(k',\underline{\mu}')N'}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$; *see Definitions* 4.1.1 *and* 4.1.3) *descends to an invertible sheaf* $\omega^{\otimes k'}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N},\mathrm{J}}$ (*resp.* $\omega^{\underline{\mu}'}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}}$, *resp.* $\omega^{\otimes(k',\underline{\mu}')}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N},\mathrm{J}}$, *resp.* $\omega^{\otimes(k',\underline{\mu}')N'}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N},\mathrm{J}}$) *over* $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$ *via the morphism* (4.2.26).

Proof. This follows from Lemma 4.2.29, as in the proof of Corollary 4.1.18. □

Proposition 4.2.31. *For each positive $\underline{\mu}'$ and each integer $k' \geq k(\underline{\mu}')$, we have canonical isomorphisms*

$$(4.2.32)\quad \begin{aligned} \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N} &\cong \mathrm{Proj}\Big(\underset{N'\geq 0}{\oplus}\Gamma(\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}, \omega^{\otimes(k',\underline{\mu}')N'}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N},\mathrm{J}})\Big) \\ &\cong \mathrm{Proj}\Big(\underset{N'\geq 0}{\oplus}\Gamma(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes(k',\underline{\mu}')N'}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}})\Big). \end{aligned}$$

This shows that $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$ is (up to canonical isomorphism) independent of the choices of $\underline{\mu}$ and the integers $k \geq k_2(\underline{\mu})$ and $N \geq N_2(\underline{\mu})$. We shall henceforth drop the subscript $(k,\underline{\mu})N$ from the notation of $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$ etc, and rewrite the morphism (4.2.26) *as a canonical morphism*

$$(4.2.33)\quad \vec{\oint}^{\mathrm{spl}}_{\mathcal{H}} : \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}.$$

Proof. By Corollary 4.2.27, $\omega^{\otimes(k',\underline{\mu}')}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ is also semiample. By Corollary 4.2.30, $\omega^{\otimes(k',\underline{\mu}')}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ descends to the invertible sheaf $\omega^{\otimes(k',\underline{\mu}')}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N},\mathrm{J}}$. Since the canonical morphism (4.2.26) is proper and surjective, the emptiness of the base locus of $\omega^{\otimes(k',\underline{\mu}')}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$ forces that of $\omega^{\otimes(k',\underline{\mu}')}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N},\mathrm{J}}$, and hence $\omega^{\otimes(k',\underline{\mu}')}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N},\mathrm{J}}$ is also semiample. Therefore, the canonical morphism $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k',\underline{\mu}')}$ factors as the composition of (4.2.26) with a canonical morphism $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N} \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k',\underline{\mu}')}$. By a symmetric argument, we also obtain a canonical morphism $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k',\underline{\mu}')} \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$, whose pre- and post-compositions with the previous canonical morphism are identity morphisms by construction. This shows that $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k,\underline{\mu})N}$ and $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H},(k',\underline{\mu}')}$ are canonically isomorphic, and that we have the canonical isomorphisms in (4.2.32), as desired. □

Proposition 4.2.34. *There is a commutative diagram*

$$(4.2.35)\quad \begin{array}{ccc} \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} & \xrightarrow{(4.2.33)} & \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}} \\ \big\downarrow{\scriptstyle \mathrm{can.}} & & \vdots \\ \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K & \xrightarrow{\vec{\oint}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K} & \vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \end{array}$$

where the dotted morphism is induced by the composition of canonical morphisms

$$\begin{aligned} \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}} &\to \mathrm{Proj}\Big(\underset{k\geq 0}{\oplus}\Gamma(\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}, \omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}},\mathrm{J}})\Big) \cong \mathrm{Proj}\Big(\underset{k\geq 0}{\oplus}\Gamma(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}})\Big) \\ &\cong \mathrm{Proj}\Big(\underset{k\geq 0}{\oplus}\Gamma(\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{J}})\Big) \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \cong \vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \end{aligned}$$

(*see Corollary* 4.2.30*, Definition* 4.1.1, [13, Prop. 7.11]*, and* [15, Thm. 6.1(2)]), *under which $\omega_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}},\mathrm{J}}$ is isomorphic to the pullback of $\omega_{\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}},\mathrm{J}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K$.*

4.3. **Main theorem for minimal compactifications.**

Theorem 4.3.1 (cf. [12, Thm. 7.2.4.1]). *For each $\mathcal{H}$ as in Choices 2.2.9, there is a normal scheme $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}$ projective and flat over $\mathrm{Spec}(\mathcal{O}_K)$, containing the scheme $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ in Definition 2.4.5 as an open fiberwise dense subscheme, such that:*

(1) *We have a commutative diagram*

$$\begin{array}{ccccc} \mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} & \twoheadrightarrow & \vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{M}^{\mathrm{min}}_{\mathcal{H}} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}} & \twoheadrightarrow & \vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \end{array} \tag{4.3.2}$$

of noetherian normal schemes flat over $\mathrm{Spec}(\mathcal{O}_K)$ and of canonical morphisms (over $\mathrm{Spec}(\mathcal{O}_K)$), in which all squares are Cartesian, all vertical arrows are open immersions with fiberwise dense image over $\mathrm{Spec}(\mathcal{O}_K)$, the two horizontal arrows at the left-hand side are open immersions with schematically dense images, the two horizontal arrows at the right-hand side are projective and surjective, and the compositions of horizontal arrows in the same rows are also open immersions with schematically dense images.

(2) *For each Σ as in Theorem 3.4.1, the commutative diagrams (3.4.2) and (4.3.2) are compatible with each other and form a commutative diagram*

$$\begin{array}{ccccc} \mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} & \twoheadrightarrow & \vec{\mathsf{M}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} & \twoheadrightarrow & \vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \\ {\scriptstyle \oint_{\mathcal{H}} \underset{F_0}{\otimes} K} \downarrow & & {\scriptstyle \vec{\oint}^{\mathrm{spl}}_{\mathcal{H}}} \downarrow & & {\scriptstyle \vec{\oint}_{\mathcal{H}}} \downarrow \\ \mathsf{M}^{\mathrm{min}}_{\mathcal{H}} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}} & \twoheadrightarrow & \vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \end{array} \tag{4.3.3}$$

in which all squares not involving $\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K$ are Cartesian, the arrows already showed up in (3.4.2) and (4.3.2) are as before, the new arrows between the bottom two rows are all proper and surjective with geometrically connected fibers, and the compositions of vertical arrows in the same columns are open immersions with fiberwise dense images.

(3) *Over $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ (resp. $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$, resp. $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}$), there is a canonical invertible sheaf $\omega^i_{\Lambda,[\tau],\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}}$ (resp. $\omega^i_{\Lambda,[\tau],\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}}$, resp. $\omega^i_{\Lambda,[\tau],\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}}$), for each $\Lambda \in \mathscr{L}$, each $[\tau] \in \Upsilon/\sim$, and each integer i satisfying $0 \leq i < d_{[\tau]}$; and there are canonical invertible sheaves $\omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$ (resp. $\omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$, resp. $\omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}},\mathrm{J}}$) and $\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$ (resp. $\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$, resp. $\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}},\mathrm{J}}$), for each integer k and each triply indexed collection of integers $\underline{\mu} = \{\mu^i_{\Lambda,[\tau]}\}_{\Lambda\in\mathscr{L}_{\mathrm{J}},[\tau]\in\Upsilon/\sim,0\leq i<d_{[\tau]}}$ that is* **positive** *in the sense that $\mu^{i-1}_{\Lambda,[\tau]} > \mu^i_{\Lambda,[\tau]}$ for all $\Lambda \in \mathscr{L}_{\mathrm{J}}$, $[\tau] \in \Upsilon/\sim$,*

and $0 < i < d_{[\tau]}$, *so that* (*cf.* (2.3.8) *and* (4.1.13))

$$\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}},\mathrm{J}} \cong \omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}},\mathrm{J}} \otimes \Big(\bigotimes_{\Lambda \in \mathscr{L}_{\mathrm{J}}} \Big(\bigotimes_{[\tau] \in \Upsilon/\sim} \Big(\bigotimes_{0 \leq i < d_{[\tau]}} (\omega^i_{\Lambda,[\tau],\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}})^{\otimes \mu^i_{\Lambda,[\tau]}} \Big) \Big) \Big),$$

$$\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}} \cong \omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}} \otimes \Big(\bigotimes_{\Lambda \in \mathscr{L}_{\mathrm{J}}} \Big(\bigotimes_{[\tau] \in \Upsilon/\sim} \Big(\bigotimes_{0 \leq i < d_{[\tau]}} (\omega^i_{\Lambda,[\tau],\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}})^{\otimes \mu^i_{\Lambda,[\tau]}} \Big) \Big) \Big),$$

and

$$\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}},\mathrm{J}} \cong \omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}},\mathrm{J}} \otimes \Big(\bigotimes_{\Lambda \in \mathscr{L}_{\mathrm{J}}} \Big(\bigotimes_{[\tau] \in \Upsilon/\sim} \Big(\bigotimes_{0 \leq i < d_{[\tau]}} (\omega^i_{\Lambda,[\tau],\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}})^{\otimes \mu^i_{\Lambda,[\tau]}} \Big) \Big) \Big).$$

Under the canonical morphisms $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ *as in Theorem* 3.4.1 *and* $\vec{\oint}^{\mathrm{spl}}_{\mathcal{H}} : \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}$ *as in* (4.3.3), *the pullbacks of the sheaves over the targets are canonical isomorphic to the corresponding sheaves* (*with similar indices*) *over the sources, while the sheaves over* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ *descend to the corresponding sheaves over* $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}$ *via the canonical morphism* $\vec{\oint}^{\mathrm{spl}}_{\mathcal{H}}$.

For each integer k, *the sheaf* $\omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$ (*resp.* $\omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$, *resp.* $\omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}},\mathrm{J}}$) *is canonically isomorphic to the pullback of the sheaf* $\omega^{\otimes k}_{\vec{\mathsf{M}}_{\mathcal{H}},\mathrm{J}}$ (*resp.* $\omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$, *resp.* $\omega^{\otimes k}_{\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}},\mathrm{J}}$) *as in* [13, Prop. 6.1 (resp. 7.11, resp. 6.4)] *and* [15, Thm. 6.1(2)]. *For all* $k > 0$, *it is semiample, and has an ample pullback to the characteristic zero fiber.*

For all positive $\underline{\mu}$, *and for all sufficiently large* k (*depending on* $\underline{\mu}$), *the sheaf* $\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}},\mathrm{J}}$ (*resp.* $\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}$, *resp.* $\omega^{\otimes(k,\underline{\mu})}_{\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}},\mathrm{J}}$) *is ample* (*resp. semiample, resp. ample*). *In particular, for all positive* $\underline{\mu}$ *and for all sufficiently large* k (*depending on* $\underline{\mu}$), *we have* $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}} \cong \mathrm{Proj}\Big(\bigoplus_{N \geq 0} \Gamma(\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}, \omega^{\otimes(k,\underline{\mu})N}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma},\mathrm{J}}) \Big)$.

(4) $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}$ *has a stratification by locally closed subschemes*

$$\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}, \tag{4.3.4}$$

with $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ *running through a complete set of cusp labels as in* [12, Def. 5.4.2.4], *such that the* $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$*-stratum* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$ *is contained in the closure of the* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$*-stratum* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *if and only if there is a surjection from the cusp label* $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$ *to the cusp label* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ *as in* [12, Def. 5.4.2.13]. *The analogous assertion holds after pullback to fibers over* $\mathrm{Spec}(\mathcal{O}_K)$.

Each $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$*-stratum* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *is flat over* $\mathrm{Spec}(\mathcal{O}_K)$ *and normal, and is canonically isomorphic to the boundary version* $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$ *of* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ (*cf. Definitions* 2.4.5 *and* 3.2.1, *and the summary in Remark* 3.2.19). *In particular,* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} = \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(0,0)]}$ *is an open fiberwise dense stratum in this stratification.*

This stratification (4.3.4) *is compatible with the stratification of* $\vec{\mathsf{M}}_{\mathcal{H}}^{\min}$ *as in* [13, Thm. 12.1 and 12.16]*; and we have a commutative diagram*

$$
\begin{array}{ccccc}
\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathsf{M}}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}} & \twoheadrightarrow & \vec{\mathsf{M}}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \\
\wr\downarrow & & \wr\downarrow & & \wr\downarrow \\
\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}^{\mathrm{spl}} & \twoheadrightarrow & \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{M}_{\mathcal{H}}^{\min} \underset{F_0}{\otimes} K & \hookrightarrow & \vec{\mathsf{M}}_{\mathcal{H}}^{\mathrm{spl},\min} & \twoheadrightarrow & \vec{\mathsf{M}}_{\mathcal{H}}^{\min} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K
\end{array}
\tag{4.3.5}
$$

of canonical morphisms, in which all squares not involving $\vec{\mathsf{M}}_{\mathcal{H}}^{\min} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathcal{O}_K$ *are Cartesian, the vertical arrows in the upper-half are isomorphisms, the vertical arrows in the bottom-half are locally closed immersions, the horizontal arrows at the left-hand sides are open immersions with schematically dense images, the horizontal arrows at the right-hand sides are projective and surjective, and the compositions of horizontal arrows in the same rows are also open immersions with schematically dense images.*

(5) *The restriction of the proper surjection* $\vec{\oint}_{\mathcal{H}}^{\mathrm{spl}}$ *in the diagram* (4.3.3) *to the stratum* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}^{\mathrm{spl}}$ *of* $\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{spl,tor}}$ *induces a surjection to the stratum* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}^{\mathrm{spl}}$ *of* $\vec{\mathsf{M}}_{\mathcal{H}}^{\mathrm{spl},\min}$*, which can be identified with the composition of the canonical isomorphism* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}^{\mathrm{spl}} \xrightarrow{\sim} \vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}^{\mathrm{spl}}$ *given by* (3.3.3) (*whose inverse appeared also in the diagram* (3.4.4))*, the structural morphism* $\vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}^{\mathrm{spl}} \to \vec{\mathsf{M}}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}$*, and the isomorphism* $\vec{\mathsf{M}}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}} \xrightarrow{\sim} \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}^{\mathrm{spl}}$ *mentioned above in* (4)*. In particular, it is proper and surjective if* σ *is top-dimensional in* $\mathbf{P}_{\Phi_{\mathcal{H}}}^{+} \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$.

Under such surjections, the commutative diagrams (3.2.20) (*expanded version*)*,* (3.4.4)*,* (4.3.5)*, and* (4.3.3) *are all compatible with each others.*

Proof. Let us take $\vec{\mathsf{M}}_{\mathcal{H}}^{\mathrm{spl},\min}$ as in Proposition 4.2.31, which is a normal scheme projective and flat over $\mathrm{Spec}(\mathcal{O}_K)$ by construction. Then, based on the corresponding assertions in [12, Thm. 7.2.4.1], the assertions (1) and (2) follow from [13, Prop. 6.1, 6.4, and 7.11] and [15, Thm. 6.1(2)], and from Propositions 4.2.25 and 4.2.34; the assertion (3) follows from [13, Prop. 6.1, 6.4, and 7.11] and [15, Thm. 6.1(2)] (again), from the definitions (see Definitions 4.1.1 and 4.1.3 and the references made from there), and from Corollary 4.2.30; and the assertions (4) and (5) follow from [13, Thm. 12.1, Cor. 12.14, and Thm. 12.16] and [15, Thm. 6.1(5)], from Proposition 4.2.15 and Corollary 4.2.18, and from the fact that the rather naive definitions [13, (6.8)] and (4.2.14) are necessarily compatible with each other. □

Corollary 4.3.6. *The canonical proper morphism*

$$\vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{spl,tor}} \to \vec{\mathsf{M}}_{\mathcal{H}}^{\mathrm{spl},\min} \underset{\vec{\mathsf{M}}_{\mathcal{H}}^{\min}}{\times} \vec{\mathsf{M}}_{\mathcal{H},\Sigma}^{\mathrm{tor}} \tag{4.3.7}$$

induced by the diagram (4.3.3) *is finite and induces a canonical isomorphism over the open dense subscheme* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$*. Consequently,* (4.3.7) *identifies its source with the normalization of its target, by Zariski's main theorem* (*see* [7, III-1, 4.4.3, 4.4.11]).

Proof. By (4) and (5) of Theorem 4.3.1, for each stratum $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ of $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ which is mapped to the stratum $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of $\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}$, the morphism

$$\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \underset{\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}{\times} \vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \tag{4.3.8}$$

induced by the pullback of (4.3.7) can be identified with the canonical morphism

$$\vec{\Xi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}} \underset{\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}}{\times} \vec{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \tag{4.3.9}$$

for any representative $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ of $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$, which is finite and induces an isomorphism from its source to the normalization of its target by Definition 3.2.3. Then (4.3.7) is quasi-finite, in particular, and hence must be finite because it is already known to be proper. When $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)] = [(0,0,\{0\}]$, (4.3.8) is just the identity morphism over $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} = \vec{\mathsf{Z}}^{\mathrm{spl}}_{[(0,0,\{0\})]}$. Thus, the corollary follows. □

Corollary 4.3.10 (cf. [13, Cor. 14.4])**.** *If the geometric fibers of* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \mathrm{Spec}(\mathcal{O}_K)$ *are reduced* (*resp. have integral local rings*)*, then all geometric fibers of* $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}} \to \mathrm{Spec}(\mathcal{O}_K)$ *have the same number of connected* (*resp. irreducible*) *components.*

Proof. As in the proof of [13, Cor. 14.4], this follows from Corollary 3.4.15, from the geometric connectedness of the fibers of $\vec{\oint}^{\mathrm{spl}}_{\mathcal{H}} : \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}$, and from the fiberwise density of $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ in $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ and $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}$ (see Theorems 3.4.1 and 4.3.1). □

Remark 4.3.11. We can improve [13, Cor. 14.4] and [15, Prop. 6.10] by assuming there that the geometric fibers of $\vec{\mathsf{M}}_{\mathcal{H}} \to \vec{\mathsf{S}}_0$ are reduced (resp. have integral local rings), by the same arguments as in the proofs of Corollaries 3.4.15 and 4.3.10.

Proposition 4.3.12 (cf. [13, Prop. 13.4, 13.9, and 13.15])**.** *With the same setting as in Proposition* 2.4.17*, the morphism* (2.4.18) *extends to a canonical projective morphism*

$$\vec{[g]}^{\mathrm{min}} : \vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}} \to \vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}'} \tag{4.3.13}$$

compatible with any morphism as in (3.4.11)*, whose pullback from* $\mathcal{O}_{F_0,(p)}$ *to* $\mathcal{O}_K$ *lifts to a canonical projective morphism*

$$\vec{[g]}^{\mathrm{spl,min}} : \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}} \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}'} \tag{4.3.14}$$

extending the morphism (2.4.19) *and compatible with any morphism as in* (3.4.12)*. The morphism* (4.3.13) (*resp.* (4.3.14)) *maps the* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$*-stratum* $\vec{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ (*resp.* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$) *of* $\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}$ (*resp.* $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}$) *to the* $[(\Phi'_{\mathcal{H}'},\delta_{\mathcal{H}'})]$*-stratum* $\vec{\mathsf{Z}}_{[(\Phi'_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$ (*resp.* $\vec{\mathsf{Z}}^{\mathrm{spl}}_{[(\Phi'_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$) *of* $\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}'}$ (*resp.* $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}'}$) *if and only if there are representatives* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ *and* $(\Phi'_{\mathcal{H}'},\delta'_{\mathcal{H}'})$ *of* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ *and* $[(\Phi'_{\mathcal{H}'},\delta'_{\mathcal{H}'})]$*, respectively, such that* $(\Phi'_{\mathcal{H}'},\delta'_{\mathcal{H}'})$ *is g-assigned to* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ *as in* [12, Def. 5.4.3.9].

Proof. These follow from the same arguments as in the proofs of Propositions 2.4.17 and 3.4.10, and from [13, Thm. 12.1 and 12.16, and Prop. 13.4], from [15, Thm. 6.1 (2) and (5)], and from (3) and (5) of Theorem 4.3.1. □

4.4. **Vanishing of higher direct images, and Koecher's principle.** By [15, Constr. 3.12 and Def. 5.13; cf. Rem. 2.9 and Cor. 5.11], we have

$$\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \cong \mathrm{NBl}_{\vec{\mathscr{J}}_{\mathcal{H},d\mathsf{pol}}}(\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}) \tag{4.4.1}$$

for some compatible collection pol of polarization functions and for some integer $d \geq 1$, for some coherent $\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}}$-ideal $\vec{\mathscr{J}}_{\mathcal{H},d\mathsf{pol}}$.

Proposition 4.4.2. *Let $\vec{\mathscr{J}}^{\mathrm{spl}}_{\mathcal{H},d\mathsf{pol}}$ denote the pullback of $\vec{\mathscr{J}}_{\mathcal{H},d\mathsf{pol}}$ to $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}$. Then we have a composition of canonical isomorphisms*

$$\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \mathrm{NBl}_{\vec{\mathscr{J}}^{\mathrm{spl}}_{\mathcal{H},d\mathsf{pol}}}(\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}) \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}} \underset{\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}}{\times} \mathrm{NBl}_{\vec{\mathscr{J}}_{\mathcal{H},d\mathsf{pol}}}(\vec{\mathsf{M}}^{\mathrm{min}}_{\mathcal{H}}), \tag{4.4.3}$$

inducing canonical isomorphisms over the common open dense subscheme $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$, which can be identified with the canonical morphism (4.3.7)*, where the first morphism is an isomorphism compatible with* (4.4.1) (*and with the canonical morphisms in* (4.3.3))*, and where the second morphism is finite and identifies its source with the normalization of its target.*

Proof. Since the (coherent ideal) pullback of $\vec{\mathscr{J}}_{\mathcal{H},d\mathsf{pol}}$ to $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is invertible, the pullback of $\vec{\mathscr{J}}^{\mathrm{spl}}_{\mathcal{H},d\mathsf{pol}}$ to $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ is also invertible. Hence, the proposition follows from the universal property of normalizations of blowups, and from Corollary 4.3.6. □

Corollary 4.4.4 (cf. [15, Cor. 6.7])**.** *There exists an effective Carter divisor D' over $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$, with $\mathsf{D}'_{\mathrm{red}} = \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} - \vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ (with its canonical reduced closed subscheme structure) such that $\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}}(-\mathsf{D}')$ is relative ample over $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}$, with respect to the canonical morphism $\vec{\oint}^{\mathrm{spl}}_{\mathcal{H},\Sigma} : \vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma} \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}$.*

Proof. This follows from [15, Cor. 6.7] and Proposition 4.4.2. □

As in [12, Sec. 7.1.2], let $\vec{\mathrm{p}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}} : \vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$ denote the structural morphism. As in [14, Sec. 6], let $\mathbf{P}^{\vee,+}_{\Phi_{\mathcal{H}}} := \{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}} : \langle \ell, y \rangle > 0, \forall y \in \mathbf{P}_{\Phi_{\mathcal{H}}} - \{0\}\}$. (We made similar definitions in [15, Sec. 8].)

Lemma 4.4.5 (cf. [15, Lem. 8.1])**.** *There exist infinitely many integers n prime to p such that, for each such n, there exists a finite étale commutative group scheme H_n of order prime to p over $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$ acting on $\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ via morphisms compatible with $\vec{\mathrm{p}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}}$, inducing canonical morphisms $\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/H_n \xrightarrow{\sim} \vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ over $\vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}},\mathrm{spl}}_{\mathcal{H}}$, whose composition we denote as $[n]$, such that $[n]^*\vec{\Psi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \cong \vec{\Psi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(n^2\ell) \cong \vec{\Psi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)^{\otimes n^2}$, for each $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}$. Moreover, for any $\mathcal{O}_K$-algebra R, the canonical morphism*

$$\vec{\Psi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathcal{O}_K}{\otimes} R \to [n]_*(\vec{\Psi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(n^2\ell) \underset{\mathcal{O}_K}{\otimes} R) \tag{4.4.6}$$

defined by adjunction identifies the left-hand side with a direct summand of the right-hand side, consisting of H_n-invariants.

Proof. This follows from [15, Lem. 8.1] and from repeated applications of Zariski's main theorem (see [7, III-1, 4.4.3, 4.4.11]), by considering the action of H_n on $\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ induced by that on $\vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, and the canonical morphism $[n] : \vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ induced by $[n] : \vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. □

By Proposition 3.2.11, $\vec{\Psi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ is isomorphic to the pullback of $\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ under the structural morphism $\vec{C}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. Therefore, by Lemma 4.4.5, and by the same arguments as in the proofs of [15, Prop. 8.3 and 8.4], we obtain the following two propositions:

Proposition 4.4.7 (cf. [15, Prop. 8.3])**.** *Suppose* $\ell \in \mathbf{P}^{\vee,+}_{\Phi_{\mathcal{H}}}$. *Then*

$$R^i(\vec{\mathrm{p}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}})_*(\vec{\Psi}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathcal{O}_K}{\otimes} R) = 0$$

for all $i > 0$ *and all* $\mathcal{O}_K$*-algebra* R.

Proposition 4.4.8 (cf. [15, Prop. 8.4])**.** *Suppose that* $\mathbf{S}_{\Phi_{\mathcal{H}}} \cong \mathbb{Z}$*, that* $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}$ *is negative, and that the morphism* $\vec{\mathrm{p}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}}$ *has positive-dimensional fibers (which is the case when the structural morphism* $\vec{\mathrm{p}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}} : \vec{C}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \vec{\mathsf{M}}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ *does). Then*

$$(\vec{\mathrm{p}}^{\mathrm{spl}}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}})_*(\vec{\Psi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathcal{O}_{F_0,(p)}}{\otimes} R) = 0$$

for all $\mathcal{O}_{F_0,(p)}$*-algebra* R.

Let R be an $\mathcal{O}_K$-algebra. Let us define the formally canonical and subcanonical quasi-coherent sheaves over $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ by the obvious analogue of [15, Def. 8.5]. By definition, the pullback of a formally canonical (resp. subcanonical) quasi-coherent sheaf over $\vec{\mathsf{M}}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ to $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ is formally canonical (resp. subcanonical). By the same arguments as in the proofs of [15, Thm. 8.6 and 8.7], with the references to [15, Thm. 6.1, and Prop. 8.3 and 8.4] there replaced with the references to Theorem 3.4.1 and Propositions 4.4.7 and 4.4.8 here, we obtain the following two theorems:

Theorem 4.4.9 (vanishing of higher direct images; cf. [14, Thm. 3.9] and [15, Thm. 8.6])**.** *Suppose* R *is an* $\mathcal{O}_K$*-algebra, and suppose that* $\mathscr{E}$ *is a quasi-coherent sheaf over* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ *that is formally canonical (resp. formally subcanonical) over* R *(as above). Let* D' *be as in Corollary* 4.4.4*, and let*

$$\mathscr{E}(-n\mathsf{D}') := \mathscr{E} \underset{\mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}}}{\otimes} \mathscr{O}_{\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}}(-n\mathsf{D}'),$$

for each integer n*. Then*

$$R^i(\vec{\oint}^{\mathrm{spl}}_{\mathcal{H},\Sigma})_*\mathscr{E}(-n\mathsf{D}') = 0$$

for all $i > 0$ *and* $n > 0$ *(resp.* $n \geq 0$*).*

Theorem 4.4.10 (Koecher's principle; compare with [14, Thm. 2.3] and [15, Thm. 8.7])**.** *Suppose* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *is a simple algebra over* $\mathbb{Q}$*. Suppose* R *is an* $\mathcal{O}_K$*-algebra, and suppose that* $\mathscr{E}$ *is a quasi-coherent sheaf over* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ *that is formally canonical over* R *(as above). For each open subset* U^{min} *of* $\vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}$*, consider its preimage* U^{tor} *in* $\vec{\mathsf{M}}^{\mathrm{spl,tor}}_{\mathcal{H},\Sigma}$ *under the canonical morphisms* $\vec{\oint}^{\mathrm{spl}}_{\mathcal{H},\Sigma}$*, and its preimage* U *in* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}}$ *under the canonical morphism* $\vec{\mathsf{M}}^{\mathrm{spl}}_{\mathcal{H}} \to \vec{\mathsf{M}}^{\mathrm{spl,min}}_{\mathcal{H}}$*. Then the canonical restriction map*

$$\Gamma(U^{\mathrm{tor}}, \mathscr{E}|_{U^{\mathrm{tor}}}) \to \Gamma(U, \mathscr{E}|_U) \tag{4.4.11}$$

is a bijection, except when $\dim(\mathsf{M}_{\mathcal{H}}) = 1$ *and* $U^{\mathrm{min}} - U \neq \emptyset$.

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