HIGHER KOECHER'S PRINCIPLE

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ABSTRACT. We show that the classical Koecher's principle for holomorphic Siegel modular forms generalizes to all PEL-type cases, in mixed characteristics and for all vector-valued weights, and also generalizes to higher coherent cohomology groups of automorphic bundles of degrees strictly less than the codimensions of the boundaries of minimal compactifications minus one.

CONTENTS

1. BACKGROUND

The classical Koecher's principle asserts that the growth condition is redundant for holomorphic Siegel modular forms *of parallel weights* and *of genus at least two*—they are automatically bounded along the cusps. In the algebro-geometric language, this means that, over a Siegel modular variety of genus at least two, the holomorphic global sections of certain *automorphic line bundles* always extend to global sections of certain *canonical extensions* of such line bundles over the toroidal compactifications. This assertion admits natural analogues for algebraic global sections (which also makes sense in mixed characteristics), for more general automorphic (vector) bundles (which are not necessarily of parallel weights), and

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for all other Shimura varieties (which generalize the Siegel modular varieties). The aim of this article is to show that these natural analogues are true, and to generalize them to the case of higher (coherent) cohomology groups of automorphic bundles.

For holomorphic automorphic forms realizing global sections of automorphic line bundles of parallel weights, the original statement in the Siegel case is [27, Satz 2], and an (earlier) analogue for Hilbert modular forms over $\mathbb{Q}(\sqrt{5})$ is in [14]. The generalization to all cases (but still for parallel weights) using Fourier–Jacobi expansions was carried out in [42, Ch. 4, Sec. 1, the proof of Lem. 2] and [5]. The methods along these lines (using q-expansions or Fourier–Jacobi expansions) have been adapted to the algebraic setup in mixed characteristics, for the global sections of powers of the Hodge line bundles, in [43, Prop. 4.9 and the paragraph following Prop. 6.6] and [10, Ch. V, Prop. 1.5].

In [32], which is a published revision of [29], a different approach was taken for the global sections of powers of the Hodge line bundles (which are automorphic line bundles of parallel weights), using the normality and boundary codimension of minimal compactifications, in all PEL-type cases. The argument there used the fact that the Hodge line bundles descend to the minimal compactifications, but the analogous statement for general automorphic bundles is not true. (There is an attempt in [12, Sec. 7] in the Siegel case, which is unfortunately based on the erroneous assertion that there is *only one* reflexive coherent sheaf extending each automorphic bundle over a toroidal compactification—when in fact there are *infinitely many*.) On the other hand, although it has been widely believed that the classical argument (using Fourier–Jacobi expansions) should also work for the global sections of all automorphic bundles, even in mixed characteristics, there has been no prior documented proof of such a belief.

In the recent works [22] and [28, Sec. 8.2], with rather different applications in mind, we showed that the higher direct images of subcanonical extensions of automorphic bundles, under the proper morphisms from toroidal compactifications to minimal compactifications, are all zero. This seemingly unrelated vanishing statement also involves in its proof an analogue of Fourier–Jacobi expansions over certain formal schemes related to the toroidal boundary, and implies an analogue of Koecher's principle even for higher cohomology groups. This led us to a *higher Koecher's principle*, which we will explain in this article.

2. Overview

This article will be built on the theory we developed earlier in [32] and [31]. We shall refer to [32] and [31] for the precise statements and for their justifications. Nevertheless, our methods also work in some other setups, and we will remark about them at the end of this article (see Section 10). (For example, our methods also work for the complex-analytically constructed compactifications of all Shimura varieties as in [3], [4], [21], and [41]. See Remark 10.1.)

Let us begin with the setup in [32]. Consider an integral model $\mathsf{M}_{\mathcal{H}} \to \mathsf{S}_0 = \operatorname{Spec}(\mathcal{O}_{F_0,(\Box)})$ of a PEL-type Shimura variety, where F_0 is the reflex field defined by the integral PEL datum $(\mathcal{O}, \star, L, \langle \cdot\,, \cdot \rangle, h_0)$ defining $\mathsf{M}_{\mathcal{H}}$, where $\Box$ is a set of good primes, where $\mathcal{O}_{F_0,(\Box)}$ is the localization of the ring of integers of F_0 with residue characteristics in $\Box$, where $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ is the integral adelic points (away from $\Box$) of the algebraic group defined by the integral PEL datum, and where $\mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ is a *neat* open compact subgroup defining the level. (See [32, Ch. 1] for more details.) Let $j^{\mathrm{tor}} : \mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ be a toroidal compactification as in [32, Thm. 6.4.1.1], and let $j^{\mathrm{min}} : \mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ be a minimal compactification as in [32, Thm. 7.2.4.1], both of which admit stratifications by locally closed subsets, compatible with each other under the canonical proper morphism $\oint_{\mathcal{H}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$. (We shall identify $\mathsf{M}_{\mathcal{H}}$ as an open subscheme of both $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ and $\mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$.) Without loss of generality, we shall assume that Σ is *projective* and smooth as in [32, Def. 7.3.1.3], to ensure that $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is a scheme projective and smooth over S_0 by [32, Thm. 7.3.3.4]. Also, with the running assumptions in [32, see, in particular, Cond. 6.2.5.25 and Thm. 6.4.1.1(3)], we can ensure that the boundary divisor $\mathsf{D}_{\infty,\mathcal{H}} := \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} - \mathsf{M}_{\mathcal{H}}$ (with its reduced subscheme structure) is a *simple* normal crossings divisor (whose irreducible components have no self-intersections).

Next let us consider the setup in [31, Sec. 6] over a base ring $R_0 := \mathcal{O}_{F_0',(\Box')}$, where F_0' is a finite extension of F_0 over which the Hodge decomposition determined by h_0 is defined, and where $\Box'$ can be taken to be any subset of $\Box$, so that we have an algebraic group scheme M_0, and so that, for any coefficient ring R over R_0 and any coefficient module $W \in \mathrm{Rep}_R(\mathrm{M}_0)$ which is locally free of finite presentation over R, we can define an automorphic bundle $\mathcal{E}_{\mathrm{M}_0,R}(W)$ over $\mathsf{M}_{\mathcal{H}}$, together with its canonical extension $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,R}(W)$ and subcanonical extension $\mathcal{E}^{\mathrm{sub}}_{\mathrm{M}_0,R}(W) = \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,R}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \mathscr{I}_{\mathsf{D}_{\infty,\mathcal{H}}}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, where $\mathscr{I}_{\mathsf{D}_{\infty,\mathcal{H}}} = \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}(-\mathsf{D}_{\infty,\mathcal{H}})$ is the $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$-ideal defining the boundary divisor $\mathsf{D}_{\infty,\mathcal{H}}$. (See [30, Rem. 5.2.14] and the references there for the relation between the algebraically and analytically defined canonical extensions when $R = \mathbb{C}$.)

For simplicity, we shall fix the choices of $\mathcal{H}$ and Σ, and often drop them from the notation. (The choice of Σ will not matter for our purpose, by the same argument as in the proof of [32, Lem. 7.1.1.4].) For example, we shall denote $\mathsf{M}_{\mathcal{H}}$ and $\oint_{\mathcal{H}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{M}^{\mathrm{min}}_{\mathcal{H}}$ by M and $\oint : \mathsf{M}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{min}}$, respectively; and denote $\mathcal{E}_{\mathrm{M}_0,R}(W)$, $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,R}(W)$, and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{M}_0,R}(W)$ by $\mathcal{E}$, $\mathcal{E}^{\mathrm{can}}$, and $\mathcal{E}^{\mathrm{sub}}$, respectively, when the context is clear.

Let $c_{\mathsf{M}} = +\infty$ if $\mathsf{M}^{\mathrm{min}} - \mathsf{M} = \emptyset$; otherwise let

$$c_{\mathsf{M}} := \mathrm{codim}(\mathsf{M}^{\mathrm{min}} - \mathsf{M}, \mathsf{M}^{\mathrm{min}}). \tag{2.1}$$

For each degree a, consider the canonical restriction morphism

$$H^a(\mathsf{M}^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}) \to H^a(\mathsf{M}, \mathcal{E}) \tag{2.2}$$

induced by $j^{\mathrm{tor}} : \mathsf{M} \hookrightarrow \mathsf{M}^{\mathrm{tor}}$.

Our first main theorem is the following generalization and reformulation of the classical Koecher's principle:

Theorem 2.3 (Koecher's principle)**.** *The morphism* (2.2) *is bijective when* $a = 0$ *and* $c_{\mathsf{M}} > 1$.

Remark 2.4. The condition $c_{\mathsf{M}} > 1$ is satisfied when every $\mathbb{Q}$-simple factor of the Shimura variety is either compact or of dimension larger than one (cf. [32, Rem. 7.2.3.15]). Thus, all noncompact curve cases are excluded. In the Siegel case this means the genus is greater than one, and in the ($\mathbb{Q}^\times$-similitude) Hilbert case this means the degree of the totally real field is greater than one.

Theorem 2.3 is a special case of the following:

Theorem 2.5 (higher Koecher's principle)**.** *The canonical morphism*

$$R^a \oint_*(\mathcal{E}^{\mathrm{can}}) \to R^a j_*^{\min} \mathcal{E} \tag{2.6}$$

over $\mathsf{M}^{\min}$ *induced by* $j^{\mathrm{tor}} : \mathsf{M} \hookrightarrow \mathsf{M}^{\mathrm{tor}}$ (*which satisfies* $\oint \circ j^{\mathrm{tor}} = j^{\min}$) *is an isomorphism for all* $a < c_{\mathsf{M}} - 1$*, and is injective for* $a = c_{\mathsf{M}} - 1$*.*

Consequently, by the Leray spectral sequence [13, Ch. II, Thm. 4.17.1]*, for each open subscheme* U *of* $\mathsf{M}^{\min}$*, the canonical restriction morphism*

$$H^a(\oint^{-1}(U), \mathcal{E}^{\mathrm{can}}) \to H^a((j^{\min})^{-1}(U), \mathcal{E}) \tag{2.7}$$

is bijective (*resp. injective*) *for all* $a < c_{\mathsf{M}} - 1$ (*resp.* $a = c_{\mathsf{M}} - 1$)*.* (*When* $U = \mathsf{M}^{\min}$*, the morphism* (2.7) *is just* (2.2).) *In particular, when* $c_{\mathsf{M}} > 1$*, every section of* $\mathcal{E}$ *over* $(j^{\min})^{-1}(U)$ (*resp.* M) *extends* (*necessarily uniquely*) *to a section of* $\mathcal{E}^{\mathrm{can}}$ *over* $\oint^{-1}(U)$ (*resp.* $\mathsf{M}^{\mathrm{tor}}$)*.*

When $R = \mathbb{C}$*, the* (*stronger*) *analogous statements are true if we replace all schemes and coherent sheaves with their complex analytifications* (*so that the sections are represented by holomorphic functions*)*, and if* U *is any open subset in the complex analytic topology.*

Such theorems are useful because one can often construct sections of $\mathcal{E}$ over M (or its complex analytification) more easily. The most typical examples are the construction of Hasse invariants and their vector-valued generalizations when R is a perfect field of some positive characteristic; and the construction of holomorphic Eisenstein series when $R = \mathbb{C}$ and when we consider the sections of the analytification of $\mathcal{E}$ over the analytification of M. A priori, these are only defined away from the boundary of $\mathsf{M}^{\mathrm{tor}}$; but Koecher's principle (as in Theorem 2.3 and its complex analytic analogue implied by Theorem 2.5) shows that, when $c_{\mathsf{M}} > 1$, such sections always extend to all of $\mathsf{M}^{\mathrm{tor}}$ (and, in the complex analytic setup, algebraizes by GAGA [44, §3, 12, Thm. 1]). Also, such theorems allow one to formulate statements in cohomological degrees strictly below $c_{\mathsf{M}} - 1$ without mentioning the toroidal compactifications at all.

Here is an outline of the rest of this article. In Section 3, we explain how to reduce the proof of Theorem 2.3 to some comparison assertion for the cohomology of certain formal schemes. We also state Theorem 3.9, which is a generalization of the vanishing theorem mentioned at the end of Section 1, and explain how to reduce its proof to some vanishing assertion for the cohomology of certain formal schemes. In Section 4, we analyze the structure of the formal schemes in question, and give sufficient conditions for the desired assertions to hold. In Sections 5, 6, and 7, we verify that the conditions are indeed satisfied, and prove Theorems 2.3 and 3.9. In Section 8, we explain how to deduce Theorem 2.5 from Theorem 8.1, the latter of which can be viewed as a stronger form of our higher Koecher's principle in the algebraic setup, and explain how to deduce Theorem 8.1 from Theorem 3.9 by Serre duality. In Section 9, we show that no higher Koecher's principle can be expected to hold in degree $c_{\mathsf{M}} - 1$, so that Theorem 2.5 is sharp. Finally, in Section 10, we remark about other setups where our methods work.

We shall follow [32, Notation and Conventions] unless otherwise specified.

3. Preliminary reductions

Since $\mathsf{M}^{\mathrm{tor}}$, M, and M_0 are separated and of finite type over S_0, and since W is locally free of finite presentation over R, by writing R as an inductive limit over its sub-R_0-algebras, we may assume that R is of finite type over R_0, which is in particular *noetherian*.

For simplicity, we shall replace S_0, M, $\mathsf{M}^{\mathrm{tor}}$, $\mathsf{M}^{\mathrm{min}}$, etc with their base changes from $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ to $\mathrm{Spec}(R)$, denoted abusively by the same symbols. (Their qualitative descriptions remain the same.) Then $\mathcal{E}$ (already defined over R, not its further base change from $\mathcal{O}_{F_0,(\Box)}$ to R) is locally free of finite rank over M, and $\mathcal{E}^{\mathrm{can}}$ and $\mathcal{E}^{\mathrm{sub}}$ are locally free of finite rank over $\mathsf{M}^{\mathrm{tor}}$.

Since the compatible collection $\Sigma = \{\Sigma_{\Phi_\mathcal{H}}\}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}$ of cone decompositions is *projective*, there exists a compatible collection $\mathsf{pol} = \{\mathsf{pol}_{\Phi_\mathcal{H}} : \mathbf{P}_{\Phi_\mathcal{H}} \to \mathbb{R}_{>0}\}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}$ of *polarization functions* (as in [32, Def. 7.3.1.1 and 7.3.1.3], based on [4, Ch. IV, Def. 2.1] and [10, Ch. V, Def. 5.1]). By [32, Thm. 6.4.1.1(3)], each irreducible component of $\mathsf{D} := \mathsf{D}_{\infty,\mathcal{H}}$ is an irreducible component of some $\overline{\mathsf{Z}}_{[(\Phi_\mathcal{H},\delta_\mathcal{H},\sigma)]}$ that is the closure of some strata $\mathsf{Z}_{[(\Phi_\mathcal{H},\delta_\mathcal{H},\sigma)]}$ labeled by the equivalence class $[(\Phi_\mathcal{H},\delta_\mathcal{H},\sigma)]$ of some triple $(\Phi_\mathcal{H},\delta_\mathcal{H},\sigma)$ such that σ is a one-dimensional cone in the cone decomposition $\Sigma_{\Phi_\mathcal{H}}$ of $\mathbf{P}_{\Phi_\mathcal{H}}$. Then we can define an effective divisor D' on $\mathsf{M}^{\mathrm{tor}}$, with $\mathsf{D}'_{\mathrm{red}} = \mathsf{D}$, such that the multiplicity of D' along each $\overline{\mathsf{Z}}_{[(\Phi_\mathcal{H},\delta_\mathcal{H},\sigma)]}$ is the value of $\mathsf{pol}_{\Phi_\mathcal{H}}$ at the $\mathbb{Z}_{>0}$-generator of $\sigma \cap \mathbf{S}^\vee_{\Phi_\mathcal{H}}$ for some (and hence every) choice of representative $(\Phi_\mathcal{H},\delta_\mathcal{H},\sigma)$. (Then $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}(-\mathsf{D}')$ is the invertible $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_\mathcal{H}}$-ideal $\mathcal{I}_{\mathcal{H},\mathsf{pol}}$ defined in [32, Thm. 7.3.3.1].)

Since the closed boundary subset $\mathsf{M}^{\mathrm{tor}} - \mathsf{M}$ of $\mathsf{M}^{\mathrm{tor}}$ is the support of the relative normal crossings divisor $\mathsf{D}'_{\mathrm{red}} = \mathsf{D}$, we have

$$j^{\mathrm{tor}}_* \mathcal{E} \cong \varinjlim_{n\geq 0} (\mathcal{E}^{\mathrm{can}}(n\mathsf{D}')) \text{ and } R^a j^{\mathrm{tor}}_* \mathcal{E} = 0 \text{ for all } a \geq 1, \tag{3.1}$$

where

$$\mathcal{E}^{\mathrm{can}}(n\mathsf{D}') := \mathcal{E}^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}}{\otimes} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}(n\mathsf{D}') \cong \mathcal{E}^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}}{\otimes} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}(-\mathsf{D}')^{\otimes(-n)}.$$

Since $j^{\mathrm{min}} = \oint \circ j^{\mathrm{tor}}$, by (3.1) (and the quasi-compactness and separateness of all schemes and morphisms involved, so that taking direct limits and direct images commute), and by the Leray spectral sequence [13, Ch. II, Thm. 4.17.1] (applied to affine open subschemes of $\mathsf{M}^{\mathrm{min}}$), for all integers a, we have

$$R^a j^{\mathrm{min}}_* \mathcal{E} \cong \varinjlim_{n\geq 0} R^a \oint_* (\mathcal{E}^{\mathrm{can}}(n\mathsf{D}')). \tag{3.2}$$

Since $\oint$ is proper and $\mathsf{M}^{\mathrm{min}}_{\mathcal{H},R}$ is noetherian, $R^a \oint_*(\mathcal{E}^{\mathrm{can}})$ and $R^a \oint_*(\mathcal{E}^{\mathrm{can}}(n\mathsf{D}'))$ are coherent for all n and a (see [18, III, 3.2.1]). In order to show that (2.6) is an isomorphism for all $a < c_\mathsf{M} - 1$ (resp. $a = 0$ when $c_\mathsf{M} > 1$), it suffices to show that

$$R^a \oint_*(\mathcal{E}^{\mathrm{can}}) \to R^a \oint_*(\mathcal{E}^{\mathrm{can}}(n\mathsf{D}')) \tag{3.3}$$

is an isomorphism for all $n \geq 0$ and all $a < c_\mathsf{M} - 1$ (resp. $a = 0$ when $c_\mathsf{M} > 1$). In order to also show that (2.6) is injective for $a = c_\mathsf{M} - 1$, it suffices to show the stronger assertion that, for all $n' \geq n \geq 0$ and all $a < c_\mathsf{M} - 1$,

$$R^a \oint_*(\mathcal{E}^{\mathrm{can}}(n'\mathsf{D}')/\mathcal{E}^{\mathrm{can}}(n\mathsf{D}')) = 0. \tag{3.4}$$

We will explain in Section 8 how to deduce this from Theorem 3.9 below.

Next let us show that the assertions in Theorem 2.5 concerning the complex analytifications follow from the corresponding algebraic assertions. Let us denote complex analytifications by the superscript "an". By the relative GAGA principle in [17, XII, Sec. 4], it suffices to prove the following:

Lemma 3.5. *The canonical morphism*

$$\big(R^a j^{\min}_* \mathcal{E}\big)^{\mathrm{an}} \to R^a (j^{\min})^{\mathrm{an}}_* (\mathcal{E}^{\mathrm{an}})$$

over $\big(\mathsf{M}^{\min}\big)^{\mathrm{an}}$ *is an isomorphism between coherent sheaves when* $a < c_{\mathsf{M}} - 1$*, and is injective* (*but not between coherent sheaves, as we shall see in the proof of Proposition* 9.1 *below*) *when* $a = c_{\mathsf{M}} - 1$.

Proof. When $c_{\mathsf{M}} = 1$, the desired assertions follow from [44, §3, 9, Prop. 10 b)]. When $c_{\mathsf{M}} > 1$, since $\mathcal{E}$ is locally free over the scheme M smooth over $\mathrm{Spec}(\mathbb{C})$, which is in particular Cohen–Macaulay (by [18, IV-4, 17.5.8]), the desired assertions follow from [16, VIII, Prop. 3.2], [17, XII, Prop. 2.1], and [47, Thm. A, A′, and B], and from the inductive argument in the proof of [47, Thm. B], the last of which shows that the canonical morphism in the last statement of [47, Thm. B] is also injective for $k = q + 2$. □

Corollary 3.6. *The canonical morphism* $H^a(\mathsf{M}, \mathcal{E}) \to H^a(\mathsf{M}^{\mathrm{an}}, \mathcal{E}^{\mathrm{an}})$ *is an isomorphism between finite* R*-modules when* $a < c_{\mathsf{M}} - 1$*, and is injective* (*but not necessarily between finite* R*-modules*) *when* $a = c_{\mathsf{M}} - 1$.

Proof. For all integers a, by (3.2) and by [44, §2, 6, Cor. 4, and Annexe, 22, Prop. 22], we have $\big(R^a j^{\min}_* \mathcal{E}\big)^{\mathrm{an}} \cong \varinjlim_{n \geq 0} \big(R^a \oint_* (\mathcal{E}^{\mathrm{can}}(n\mathsf{D}'))\big)^{\mathrm{an}}$, an inductive limit of coherent sheaves over the projective $\big(\mathsf{M}^{\min}\big)^{\mathrm{an}}$, to which GAGA [44, §3, 12, Thm. 1] also applies. Hence the corollary follows from Lemma 3.5 and the Leray spectral sequence [13, Ch. II, Thm. 4.17.1], as desired. □

Remark 3.7. In the proof of [6, Thm. 10.14], they used [45, Thm. 1 and 2] and [44, §3, 12, Thm. 1] to show that $(j^{\min})^{\mathrm{an}}_*(\mathcal{E}^{\mathrm{an}})$ is coherent and algebraizable by $j^{\min}_* \mathcal{E}$ when $c_{\mathsf{M}} > 1$. It is worth nothing that the question raised at the end of [45] has been addressed by [48] and [47].

So we will no longer see complex analytifications in our methods. Our first goal will be to prove Theorem 2.3. Although Theorem 2.3 is a special case of Theorem 2.5, it will be proved by a method that also generalizes to the case of partial compactifications in [28], which might be of some independent interest.

By the reduction steps above, in order to prove Theorem 2.3, it suffices to show that, for an arbitrary geometric point $\bar{x}$ of $\mathsf{M}^{\min}$, the pullback of (3.3) to the completion $(\mathsf{M}^{\min})^{\wedge}_{\bar{x}}$ of the strict localization of $\mathsf{M}^{\min}$ at $\bar{x}$ is an isomorphism when $a = 0$ and $c_{\mathsf{M}} > 1$. Let us fix any such $\bar{x}$, which lies on some stratum $\mathsf{Z} = \mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ of $\mathsf{M}^{\min}$ with some choice of a representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of the cusp label $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$; and let us denote by $(\,\cdot\,)^{\wedge}_{\bar{x}}$ the pullback of any $(\,\cdot\,)$ under the canonical morphism $(\mathsf{M}^{\min})^{\wedge}_{\bar{x}} \to \mathsf{M}^{\min}$. We may and we shall assume that Z is a strata of $\mathsf{M}^{\min} - \mathsf{M}$, because D' is supported on $\mathsf{M}^{\mathrm{tor}} - \mathsf{M} = \oint^{-1}(\mathsf{M}^{\min} - \mathsf{M})$. (Our theorems are trivially true when $\mathsf{M}^{\mathrm{tor}} - \mathsf{M} = \emptyset$.) Again since $\oint$ is proper and $\mathsf{M}^{\min}_{\mathcal{H},R}$ is noetherian, by [18, III-1, 4.1.5], it suffices to show that, for all $n > 0$, the canonical morphism

$$H^0((\mathsf{M}^{\mathrm{tor}})^{\wedge}_{\bar{x}}, (\mathcal{E}^{\mathrm{can}})^{\wedge}_{\bar{x}}) \to H^0((\mathsf{M}^{\mathrm{tor}})^{\wedge}_{\bar{x}}, (\mathcal{E}^{\mathrm{can}}(n\mathsf{D}))^{\wedge}_{\bar{x}}) \tag{3.8}$$

is bijective when $c_{\mathsf{M}} > 1$.

Our second goal will be to prove the following:

Theorem 3.9. $R^a \oint_* \mathcal{E}^{\mathrm{can}}(-n'\mathsf{D}') = R^a \oint_* \mathcal{E}^{\mathrm{sub}}(-n\mathsf{D}') = R^a \oint_* \mathcal{E}^{\mathrm{sub}} = 0$ *for all* $n' > n \geq 0$ *and all* $a > 0$.

Remark 3.10. The assertion that $R^a \oint_* \mathcal{E}^{\rm sub} = 0$ for all $a > 0$ was proved for certain unitary cases in [22] and for all PEL-type cases in [28, Sec. 8.2]. The proof of Theorem 3.9 will be essentially a review of the one in [28, Sec. 8.2], based on ideas relating nerves of certain coverings of the boundary to the cone decompositions defining them, which can be traced back to the seminal works [26, Ch. I, Sec. 3] and [23]. (Under more restrictive assumptions, the assertion was also proved in [33] by a rather different method, using crucially the automorphic vanishing in [36, Thm. 8.13(2)] based on Kodaira-type vanishing results. With a different setup on the levels and weights, the analogous assertions were proved for the Hilbert and Siegel cases in [1] and [2] by yet another method. See the introduction of [33] for a more detailed overview.) While we will only need the assertion $R^a \oint_* \mathcal{E}^{\rm sub} = 0$ for proving Theorem 2.5 (see Remark 8.11 below), Theorem 3.9 will nevertheless allow us to also prove Theorem 8.1, which can be viewed as a stronger form of our higher Koecher's principle. Also, presenting the proof of Theorem 3.9 will make it clear that the methods also work in many other setups (see Section 10).

By the reduction steps above, in order to prove Theorem 3.9, it suffices to show that, for all $n' > n \geq 0$ and all $a > 0$, we have

$$H^a((\mathsf{M}^{\rm tor})^\wedge_{\bar{x}}, (\mathcal{E}^{\rm can}(-n'\mathsf{D}'))^\wedge_{\bar{x}}) \xrightarrow{\sim} H^a((\mathsf{M}^{\rm tor})^\wedge_{\bar{x}}, (\mathcal{E}^{\rm sub}(-n\mathsf{D}'))^\wedge_{\bar{x}}) = 0 \tag{3.11}$$

These two goals will be achieved in the next four sections.

4. Cohomology of certain formal schemes

The aim of this section is to describe the morphism $\oint^\wedge_{\bar{x}} : (\mathsf{M}^{\rm tor})^\wedge_{\bar{x}} \to (\mathsf{M}^{\rm min})^\wedge_{\bar{x}}$, to give sufficient conditions for (3.8) to be an isomorphism when $c_{\mathsf{M}} > 1$, and to give sufficient conditions for (3.11) to hold for all $n' > n \geq 0$ and all $a > 0$. (The setup will be similar to the one in [31, Sec. 3–4].)

First let us consider the formal completion $(\mathsf{M}^{\rm min})^\wedge_{\mathsf{Z}}$ of $\mathsf{M}^{\rm min}$ along Z. (Since Z is locally closed, we remove the other strata of $\mathsf{M}^{\rm min}$ in the closure of Z before taking the formal completion.) We shall denote by $(\,\cdot\,)^\wedge_{\mathsf{Z}}$ the pullback of any $(\,\cdot\,)$ under the canonical morphism $(\mathsf{M}^{\rm min})^\wedge_{\mathsf{Z}} \to \mathsf{M}^{\rm min}$.

By [32, Thm. 6.4.1.1(5) and Thm. 7.2.4.1(5)], and by the same argument as in the proof of [31, Prop. 4.3], using [32, Thm. 6.4.1.1(6)], the formal scheme $(\mathsf{M}^{\rm tor})^\wedge_{\mathsf{Z}}$ admits an open covering by formal schemes $\mathfrak{U}_{[\sigma]}$ parameterized by orbits $[\sigma]$ in $\Sigma^+_{\Phi_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}}}$, where

$$\Sigma^+_{\Phi_{\mathcal{H}}} := \{\sigma \in \Sigma_{\Phi_{\mathcal{H}}} : \sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}\}.$$

Since the choice of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ has been fixed, so that we will not need the full collections Σ and pol, we shall write Γ, Σ, Σ^+, $\mathbf{S}$, $\mathbf{S}^\vee$, $\mathbf{S}^\vee_{\mathbb{R}}$, $\mathbf{P}$, $\mathbf{P}^+$, and pol instead of $\Gamma_{\Phi_{\mathcal{H}}}$, $\Sigma_{\Phi_{\mathcal{H}}}$, $\Sigma^+_{\Phi_{\mathcal{H}}}$, $\mathbf{S}_{\Phi_{\mathcal{H}}}$, $\mathbf{S}^\vee_{\Phi_{\mathcal{H}}}$, $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$, $\mathbf{P}_{\Phi_{\mathcal{H}}}$, $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$, and $\mathsf{pol}_{\Phi_{\mathcal{H}}}$, respectively.

For each $\sigma \in \Sigma^+$ representing $[\sigma]$, we have a canonical isomorphism $\mathfrak{U}_{[\sigma]} \cong \mathfrak{U}_\sigma$, where the structure of $\mathfrak{U}_\sigma$ can be described as follows:

(1) There is an abelian scheme torsor C (denoted $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ in [32, Sec. 6.2.4]) over a finite étale covering of Z, carrying an action of Γ over Z.

(2) There is a torus torsor $\Xi \cong \underline{\rm Spec}_{\mathscr{O}_C}\Big(\bigoplus_{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}} \Psi(\ell)\Big) \to C$ (denoted $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ in [32, Sec. 6.2.4]) under the split torus E (denoted $E_{\Phi_{\mathcal{H}}}$ in [32, Sec. 6.2.4]) with character group $\mathbf{S}$, given by a group homomorphism $\Psi : \mathbf{S} \to \underline{\rm Pic}(C) : \ell \mapsto \Psi(\ell)$ (denoted $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ in [32, Sec. 6.2.4]).

(3) Each cone $\tau \in \Sigma$, which is a subset of $\mathbf{P} \subset \mathbf{S}^\vee_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}, \mathbb{R})$, defines an affine toroidal embedding $\Xi \hookrightarrow \Xi(\tau) := \underline{\mathrm{Spec}}_{\mathscr{O}_C}\Big(\underset{\ell\in\tau^\vee}{\oplus}\Psi(\ell)\Big)$ over C, with its closed τ-stratum given by $\Xi_\tau := \underline{\mathrm{Spec}}_{\mathscr{O}_C}\Big(\underset{\ell\in\tau^\perp}{\oplus}\Psi(\ell)\Big)$ defined by the $\mathscr{O}_{\Xi(\tau)}$-ideal given by the sub-$\mathscr{O}_C$-module $\underset{\ell\in\tau_0^\vee}{\oplus}\Psi(\ell)$ of $\underset{\ell\in\tau^\vee}{\oplus}\Psi(\ell)$, where

$$\begin{aligned}\tau^\vee &:= \{\ell \in \mathbf{S} : \langle \ell, y\rangle \geq 0 \ \forall y \in \tau\},\\ \tau_0^\vee &:= \{\ell \in \mathbf{S} : \langle \ell, y\rangle > 0 \ \forall y \in \tau\}, \text{ and}\\ \tau^\perp &:= \{\ell \in \mathbf{S} : \langle \ell, y\rangle = 0 \ \forall y \in \tau\} \cong \tau^\vee/\tau_0^\vee\end{aligned}$$

are semisubgroups of $\mathbf{S}$.

(4) Each cone $\sigma \in \Sigma^+$, which is a subset of $\mathbf{P}^+ \subset \mathbf{P}$, defines a closed subscheme U_σ of $\Xi(\sigma)$ given by the union of Ξ_τ for all faces τ of σ that are also in Σ^+, which is defined by the $\mathscr{O}_{\Xi(\tau)}$-ideal given by the $\mathscr{O}_C$-submodule $\underset{\ell\in\sigma_{0+}^\vee}{\oplus}\Psi(\ell)$ of $\underset{\ell\in\tau^\vee}{\oplus}\Psi(\ell)$, where $\sigma_{0+}^\vee := \underset{\tau\in\Sigma^+,\tau\subset\overline{\sigma}}{\cap}\tau_0^\vee$ (where the intersection takes place in $\mathbf{S}$, where $\overline{\sigma}$ denotes the closure of σ in $\mathbf{S}^\vee_{\mathbb{R}}$, and where $\tau = \sigma$ is included).

(5) Then $\mathfrak{U}_\sigma$ is the formal completion of $\Xi(\sigma)$ along its closed subscheme U_σ. By abuse of notation, we write

$$\mathscr{O}_{\mathfrak{U}_\sigma} \cong \underset{\ell\in\sigma^\vee}{\hat{\oplus}}\Psi(\ell), \tag{4.1}$$

where $\underset{\ell\in\sigma^\vee}{\hat{\oplus}}\Psi(\ell)$ denotes the formal completion of $\underset{\ell\in\sigma^\vee}{\oplus}\Psi(\ell)$ with respect to the topology defined by the $\mathscr{O}_C$-submodule $\underset{\ell\in\sigma_{0+}^\vee}{\oplus}\Psi(\ell)$.

(6) For $\sigma, \tau \in \Sigma^+$ such that τ is a face of σ, we have $\sigma^\vee \subset \tau^\vee$ and $\sigma_{0+}^\vee \subset \tau_{0+}^\vee$ by definition, and we have a canonical open immersion $\mathfrak{U}_\tau \hookrightarrow \mathfrak{U}_\sigma$ induced by the canonical open immersion $\Xi(\tau) \hookrightarrow \Xi(\sigma)$ over C. The morphism $\mathfrak{U}_\tau \hookrightarrow \mathfrak{U}_\sigma$ of formal schemes relatively affine over C corresponds to the $\mathscr{O}_C$-algebra homomorphism $\underset{\ell\in\sigma^\vee}{\hat{\oplus}}\Psi(\ell) \to \underset{\ell\in\tau^\vee}{\hat{\oplus}}\Psi(\ell)$, which induces the identity morphism on $\Psi(\ell)$ when $\ell \in \sigma^\vee$.

(7) For each $\gamma \in \Gamma$, we have a canonical isomorphism

$$\gamma : \mathfrak{U}_\sigma \xrightarrow{\sim} \mathfrak{U}_{\gamma\sigma} \tag{4.2}$$

covering the canonical isomorphism $\gamma : C \xrightarrow{\sim} C$ over Z given by the action of Γ on $C \to \mathsf{Z}$, induced by the isomorphisms

$$\gamma^* : \gamma^*\Psi(\gamma\ell) \xrightarrow{\sim} \Psi(\ell) \tag{4.3}$$

over C.

(8) The formal schemes $\{\mathfrak{U}_\sigma\}_{\sigma\in\Sigma^+}$ relative affine over C glue together and form a formal scheme $\mathfrak{X}$ over C. The isomorphisms $\gamma : \mathfrak{U}_\sigma \xrightarrow{\sim} \mathfrak{U}_{\gamma\sigma}$ induce an action of Γ on $\mathfrak{X}$ covering the action of Γ on $C \to \mathsf{Z}$, which is free by the neatness of $\mathcal{H}$ and by [32, Cond. 6.2.5.25 and Lem. 6.2.5.27], which induces a *local isomorphism*

$$\mathfrak{X} \to \mathfrak{X}/\Gamma. \tag{4.4}$$

Moreover, by the same argument as in the proof of [31, Prop. 4.3], based on [32, Thm. 6.4.1.1(6)], there is a canonical isomorphism

$$\mathfrak{X}/\Gamma \cong (\mathsf{M}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}} \tag{4.5}$$

of formal schemes, inducing a canonical morphism $(\mathsf{M}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}} \to \mathsf{Z}$.

The pullback of the above $(\mathsf{M}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}} \to \mathsf{Z}$ under the canonical morphism $\mathsf{Z}^{\wedge}_{\bar{x}} \to \mathsf{Z}$ is the canonical morphism $(\mathsf{M}^{\mathrm{tor}})^{\wedge}_{\bar{x}} \to \mathsf{Z}^{\wedge}_{\bar{x}}$ given by the composition of the canonical morphism $(\mathsf{M}^{\mathrm{tor}})^{\wedge}_{\bar{x}} \to (\mathsf{M}^{\mathrm{min}})^{\wedge}_{\bar{x}}$ with the *structural morphism* $(\mathsf{M}^{\mathrm{min}})^{\wedge}_{\bar{x}} \to \mathsf{Z}^{\wedge}_{\bar{x}}$ as in [32, Prop. 7.2.3.16]. Thus, we have an open covering of $(\mathsf{M}^{\mathrm{tor}})^{\wedge}_{\bar{x}}$ by formal schemes

$$\mathfrak{V}_{[\sigma]} := (\mathfrak{U}_{[\sigma]})^{\wedge}_{\bar{x}} := \mathfrak{U}_{[\sigma]} \underset{\mathsf{Z}}{\times} \mathsf{Z}^{\wedge}_{\bar{x}}$$

parameterized by the Γ-orbits $[\sigma]$ in Σ^+/Γ, each $\mathfrak{V}_{[\sigma]}$ being canonically isomorphic to

$$\mathfrak{V}_{\sigma} := (\mathfrak{U}_{\sigma})^{\wedge}_{\bar{x}} := \mathfrak{U}_{\sigma} \underset{\mathsf{Z}}{\times} \mathsf{Z}^{\wedge}_{\bar{x}}$$

when $\sigma \in \Sigma^+$ is a representative of the Γ-orbit $[\sigma]$. Accordingly, we have an open covering of $\mathfrak{Y} := \mathfrak{X}^{\wedge}_{\bar{x}}$ by the formal schemes $\mathfrak{V}_{\sigma}$, and the local isomorphism

$$\mathfrak{Y} \to \mathfrak{Y}/\Gamma \tag{4.6}$$

induced by (4.4) and the isomorphism

$$\mathfrak{Y}/\Gamma \cong (\mathsf{M}^{\mathrm{tor}})^{\wedge}_{\bar{x}} \tag{4.7}$$

induced by (4.5) are compatible with the isomorphisms

$$\gamma : \mathfrak{V}_{\sigma} \xrightarrow{\sim} \mathfrak{V}_{\gamma\sigma}, \tag{4.8}$$

for $\gamma \in \Gamma$, induced by (4.2). (For simplicity, we shall write γ instead of $\gamma^{\wedge}_{\bar{x}}$.) By abuse of language, we shall say that $\mathfrak{V}_{\sigma}$ is relative affine over the (relative) abelian scheme $C^{\wedge}_{\bar{x}}$ over $\mathsf{Z}^{\wedge}_{\bar{x}}$, and write

$$\mathscr{O}_{\mathfrak{V}_{\sigma}} \cong \underset{\ell \in \sigma^{\vee}}{\hat{\oplus}} (\Psi(\ell))^{\wedge}_{\bar{x}}. \tag{4.9}$$

as in (4.1). Then the isomorphism (4.8) is induced by the isomorphisms

$$\gamma^* : \gamma^*(\Psi(\gamma\ell))^{\wedge}_{\bar{x}} \xrightarrow{\sim} (\Psi(\ell))^{\wedge}_{\bar{x}} \tag{4.10}$$

induced by (4.3).

Definition 4.11. *For each integer m, we denote by $\mathscr{E}^{(m)}$ (resp. $\mathscr{E}^{(m)+}$) the pullback of $\mathcal{E}^{\mathrm{can}}(m\mathsf{D}')$ (resp. $\mathcal{E}^{\mathrm{sub}}(m\mathsf{D}')$) to $\mathfrak{Y}/\Gamma$ under the composition*

$$\mathfrak{Y}/\Gamma \overset{(4.7)}{\to} (\mathsf{M}^{\mathrm{tor}})^{\wedge}_{\bar{x}} \overset{\mathrm{can.}}{\to} \mathsf{M}^{\mathrm{tor}}.$$

We shall abusively denote by the same symbols its further pullback to $\mathfrak{Y}$ under (4.6)*. When $m = 0$, we shall simply denote the pullbacks of $\mathcal{E}^{\mathrm{can}}$ by $\mathscr{E}$.*

Consider the union $\widetilde{\mathfrak{N}}$ of the cones σ in Σ^+, which admits a closed covering by the closures σ^{cl} (in $\widetilde{\mathfrak{N}}$) of the cones σ in Σ^+ (with natural incidence relations inherited from those of the cones σ as locally closed subsets of $\mathbf{S}^{\vee}_{\mathbb{R}}$). Let

$$\mathfrak{N} := \widetilde{\mathfrak{N}}/\Gamma.$$

By definition, the nerve of the open covering $\{\mathfrak{V}_{\sigma}\}_{\sigma \in \Sigma^+}$ of $\mathfrak{Y}$ is naturally identified with the nerve of the (locally finite) closed covering $\{\sigma^{\mathrm{cl}}\}_{\sigma \in \Sigma^+}$ of $\widetilde{\mathfrak{N}}$. Accordingly, the nerve of the open covering $\{\mathfrak{V}_{[\sigma]}\}_{[\sigma] \in \Sigma^+/\Gamma}$ of $\mathfrak{Y}/\Gamma \cong (\mathsf{M}^{\mathrm{tor}})^{\wedge}_{\bar{x}}$ (see (4.7)) is naturally identified with the nerve of the (finite) closed covering $\{[\sigma]^{\mathrm{cl}}\}_{[\sigma]\in\Sigma^+/\Gamma}$ of $\mathfrak{N}$, where $[\sigma]^{\mathrm{cl}}$ denotes the closure of $[\sigma]$ in $\mathfrak{N}$.

Note that the assumption that Z is a stratum of $\mathsf{M}^{\min} - \mathsf{M}$ means $\Sigma^+ \neq \emptyset$ and $\widetilde{\mathfrak{N}} = \mathbf{P}^+ \neq \emptyset$.

Suppose $\mathscr{M}$ is a quasi-coherent sheaf over $\mathfrak{Y}/\Gamma$. For each $d \in \mathbb{Z}$, consider the constructible sheaf $\underline{\mathscr{H}}^d(\mathscr{M})$ over $\mathfrak{N}$ which has stalks $H^d(\mathfrak{V}_{[\sigma]}\mathscr{M}|_{\mathfrak{U}_{[\sigma]}})$ over $[\sigma]$, where $[\sigma] \in \Sigma^+/\Gamma$ is viewed as a locally closed stratum of $\mathfrak{N}$. Then we have the following spectral sequence (based on [13, Ch. II, 5.2.1, 5.2.4, and 5.4.1]):

$$E_2^{c,d} := H^c(\mathfrak{N}, \underline{\mathscr{H}}^d(\mathscr{M})) \Rightarrow H^{c+d}(\mathfrak{Y}/\Gamma, \mathscr{M}). \tag{4.12}$$

By abuse of notation, let us also denote by $\mathscr{M}$ its pullback to $\mathfrak{Y}$, and denote by $\underline{\mathscr{H}}^d(\mathscr{M})$ its pullback to $\widetilde{\mathfrak{N}}$. Since (4.6) is a local isomorphism, the E_2 terms of (4.12) can be computed by the spectral sequence

$$E_2^{c-e,e} := H^{c-e}(\Gamma, H^e(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^d(\mathscr{M}))) \Rightarrow H^c(\mathfrak{N}, \underline{\mathscr{H}}^d(\mathscr{M})). \tag{4.13}$$

These spectral sequences are canonical and functorial in $\mathscr{M}$. Hence we have:

Lemma 4.14. *Given any integer $n \geq 0$, in order to show that* (3.8) *is bijective when $c_{\mathsf{M}} > 1$, it suffices to show that the canonical morphism*

$$H^0(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^0(\mathscr{E})) \to H^0(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^0(\mathscr{E}^{(n)})) \tag{4.15}$$

is (Γ*-equivariant and*) *bijective when $c_{\mathsf{M}} > 1$.*

Lemma 4.16. *Given any integers $n' > n \geq 0$, in order to show that* (3.11) *holds for all $a > 0$, it suffices to show the following:*

(1) $H^e(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^d(\mathscr{E}^{(-n')})) \xrightarrow{\sim} H^e(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^d(\mathscr{E}^{(-n)+})) = 0$ for all $d \geq 0$ and $e > 0$.

(2) $H^c(\Gamma, H^0(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^d(\mathscr{E}^{(-n')}))) \xrightarrow{\sim} H^c(\Gamma, H^0(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^d(\mathscr{E}^{(-n)+}))) = 0$ when either $c > 0$ or $d > 0$.

Remark 4.17. By the arguments in [26, Ch. I, Sec. 3, especially Cor. 2], for proving Lemmas 4.14 and 4.16, we may replace the cone decomposition Σ with locally finite refinements (without modifying pol) which are still smooth; also, for proving (1) of Lemma 4.16, we may replace Σ with refinements which are not necessarily Γ-invariant. (When doing so, we may have to give up the condition in the definition of pol, as in [32, Thm. 7.3.1.1(3)], that pol is linear on a subset of $\mathbf{P}$ *if and only if* the subset is contained in some cone in Σ. Such a condition will not be needed in the arguments below. What we will need is that pol is linear on each cone in Σ, which is unaffected by refining Σ.)

5. Pullbacks of canonical extensions

Let us describe the pullbacks of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}(n\mathsf{D}')$ and $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}(-\mathsf{D} + n\mathsf{D}')$ to $\mathfrak{U}_\sigma$. By definition, for each one-dimensional cone $\tau \in \Sigma$ that is a face of a cone $\sigma \in \Sigma^+$, the closure of the τ-stratum of $\Xi(\sigma)$ is defined by the $\mathscr{O}_{\Xi(\sigma)}$-ideal given by the sub-$\mathscr{O}_C$-module $\underset{\ell\in\tau_0^\vee\cap\sigma^\vee}{\oplus}\Psi(\ell)$ of $\underset{\ell\in\sigma^\vee}{\oplus}\Psi(\ell)$. Suppose s_τ is a $\mathbb{Z}_{>0}$-generator of the semigroup $\tau\cap\mathbf{S}^\vee$. Then we have

$$\tau_0^\vee = \{\ell \in \mathbf{S} : \langle \ell, s_\tau\rangle > 0\} = \{\ell \in \mathbf{S} : \langle \ell, s_\tau\rangle \geq 1\}.$$

To achieve multiplicity $\mathsf{pol}(s_\tau)$ on the closure of the τ-stratum, as in the definition of the divisor D' on $\mathsf{M}^{\mathrm{tor}}$, we need $\langle \ell, s_\tau \rangle \geq \mathsf{pol}(s_\tau)$, or rather

$$\langle \ell, y \rangle \geq \mathsf{pol}(y) \tag{5.1}$$

for all $y \in \tau$. Since σ is the $\mathbb{R}_{>0}$-span of its one-dimensional faces, and since pol is a linear function on σ, the condition that (5.1) holds for all y in the one-dimensional faces of σ is equivalent to the condition that (5.1) holds for all y in σ. Let us define, for each $n \in \mathbb{Z}$, the following subsets of $\mathbf{S}$:

$$\sigma^\vee_{(n)} := \{\ell \in \mathbf{S} : \langle \ell, y \rangle \geq -n\mathsf{pol}(y) \ \forall y \in \sigma\} \tag{5.2}$$

and

$$\sigma^\vee_{(n)+} := \{\ell \in \mathbf{S} : \langle \ell, y \rangle > -n\mathsf{pol}(y) \ \forall y \in \sigma\}. \tag{5.3}$$

Lemma 5.4. *For each cone $\sigma = \mathbb{R}_{>0}v_1 + \cdots + \mathbb{R}_{>0}v_r$ in Σ^+, where $v_1, \ldots, v_r$ are nonzero rational vectors in $\mathbf{P}$, we have the following criteria:*

(1) $\ell \in \sigma^\vee_{(n)}$ if and only if $\langle \ell, v_i \rangle \geq -n\mathsf{pol}(v_i)$ for all $1 \leq i \leq r$.
(2) $\ell \in \sigma^\vee_{(n)+}$ if and only if $\langle \ell, v_i \rangle > -n\mathsf{pol}(v_i)$ for all $1 \leq i \leq r$.

Proof. Since $-n\mathsf{pol}$ is a linear function on σ for each $n \in \mathbb{Z}$, these follow immediately from the definitions (see (5.2) and (5.3)). $\square$

Hence, the $\mathscr{O}_{\Xi(\sigma)}$-ideals defining the pullback of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}(-\mathsf{D})$ (resp. $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}(-\mathsf{D}'))$ to $\mathfrak{U}_\sigma$ is given by the sub-$\mathscr{O}_C$-module $\hat{\bigoplus}_{\ell \in \sigma^\vee_{(0)+}} \Psi(\ell)$ (resp. $\hat{\bigoplus}_{\ell \in \sigma^\vee_{(-1)}} \Psi(\ell)$) of $\hat{\bigoplus}_{\ell \in \sigma^\vee} \Psi(\ell)$. More generally, let $\mathscr{O}^{(n)}_{\mathfrak{U}_\sigma}$ (resp. $\mathscr{O}^{(n)+}_{\mathfrak{U}_\sigma}$) denote the pullback of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}(n\mathsf{D}')$ (resp. $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}(-\mathsf{D} + n\mathsf{D}'))$ to $\mathfrak{U}_\sigma$, and by $\mathscr{O}^{(n)}_{\mathfrak{V}_\sigma}$ (resp. $\mathscr{O}^{(n)+}_{\mathfrak{V}_\sigma}$) the pullback of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}(n\mathsf{D}')$ (resp. $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}(-\mathsf{D} + n\mathsf{D}'))$ to $\mathfrak{V}_\sigma$. Then, by abuse of language, for ? = $\emptyset$ or $+$, we have $\mathscr{O}^{(n)?}_{\mathfrak{U}_\sigma} \cong \hat{\bigoplus}_{\ell \in \sigma^\vee_{(n)?}} \Psi(\ell)$ and

$$\mathscr{O}^{(n)?}_{\mathfrak{V}_\sigma} \cong \hat{\bigoplus}_{\ell \in \sigma^\vee_{(n)?}} (\Psi(\ell))^\wedge_{\bar{x}}. \tag{5.5}$$

Proposition 5.6. *The pullback of $\mathscr{E}^{\mathrm{can}}$ to $\mathfrak{X}$ under the composition*

$$\mathfrak{X} \stackrel{(4.4)}{\to} \mathfrak{X}/\Gamma \stackrel{(4.5)}{\to} (\mathsf{M}^{\mathrm{tor}})^\wedge_{\mathsf{Z}} \stackrel{\mathrm{can.}}{\to} \mathsf{M}^{\mathrm{tor}}$$

is canonically isomorphic to the pullback of a canonically determined locally free coherent sheaf $\mathscr{E}_0$ over C under the canonical morphism $\mathfrak{X} \to C$. Moreover, $\mathscr{E}_0$ admits a filtration such that its graded pieces are isomorphic to pullbacks of coherent sheaves over Z that are locally over Z isomorphic to pullbacks of coherent sheaves over $\mathsf{M}_0 = \mathrm{Spec}(R)$. (However, since Γ acts on C, this does not imply that the pullback of $\mathscr{E}^{\mathrm{can}}$ to $\mathfrak{X}/\Gamma$ admits such a filtration.)

Proof. Let $(G, G^\vee, \lambda : G \to G^\vee) \to \mathsf{M}^{\mathrm{tor}}$ denote the tautological semi-abelian scheme, the dual semi-abelian scheme, and the homomorphism whose pullbacks under $j^{\mathrm{tor}} : \mathsf{M} \hookrightarrow \mathsf{M}^{\mathrm{tor}}$ define the tautological abelian scheme, its dual abelian scheme, and the tautological (separable) polarization over M. Then it is part of the construction of the isomorphism (4.5) that the pullback of $(G, G^\vee, \lambda : G \to G^\vee)$ to $\mathfrak{X}$ (as formal schemes, not as relative schemes) canonically descends to a similar tuple $(G^\natural, G^{\vee,\natural}, \lambda^\natural : G^\natural \to G^{\vee,\natural})$ over C.

As explained in [31, Sec. 6B] (with all objects pulled back to R, and with (1) denoting formal Tate twists), there exists a projective $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-module L_0 such that M_0 is the automorphism group scheme of $(L_0^\vee(1) \underset{R}{\otimes} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}(1))$ (in the obvious sense), and such that

$$\mathcal{E}^{\mathrm{can}} = \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,R}(W) = \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0} \overset{\mathrm{M}_0}{\times} W,$$

where

$$\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}}\big((\underline{\mathrm{Lie}}^\vee_{G^\vee/\mathsf{M}^{\mathrm{tor}}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}(1)), (L_0^\vee(1), R(1)) \underset{R}{\otimes} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}\big)$$

is the canonical extension over $\mathsf{M}^{\mathrm{tor}}$ of the principle M_0-bundle over M. Then we can verify the first assertion of the proposition by similarly defining

$$\mathcal{E}_C := \mathcal{E}_{\mathrm{M}_0,C} \overset{\mathrm{M}_0}{\times} W,$$

where

$$\mathcal{E}_{\mathrm{M}_0,C} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_C}\big((\underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/C}, \mathscr{O}_C(1)), (L_0^\vee(1), R(1)) \underset{R}{\otimes} \mathscr{O}_C\big)$$

(cf. [31, Lem. 6.16] and its proof).

Moreover, we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T & \longrightarrow & G^\natural & \longrightarrow & B & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \lambda_T} & & \downarrow{\scriptstyle \lambda^\natural} & & \downarrow{\scriptstyle \lambda_B} & & \\ 1 & \longrightarrow & T^\vee & \longrightarrow & G^{\vee,\natural} & \longrightarrow & B^\vee & \longrightarrow & 1 \end{array}$$

of semi-abelian schemes over C, with columns being separable isogenies and rows being exact, inducing a similar commutative diagram between their relative Lie algebras, with columns being isomorphisms and rows being exact, such that the torus part $(T, T^\vee, \lambda_T)$ and the abelian part $(B, B^\vee, \lambda_B)$ both descend to Z, which we abusively denote by the same symbols. Accordingly, there exists a parabolic subgroup scheme $\mathrm{P}_{0,\mathsf{Z}}$ of M_0, which is the stabilizer of a projective $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-module quotient $L_{0,\mathsf{Z}}$ of L_0, such that, by the same argument as in the proof of [35, Lem. 1.19], the canonical forgetful morphism from

$$\mathcal{E}_{\mathrm{P}_{0,\mathsf{Z}}} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_C}\big((\underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/C}, \underline{\mathrm{Lie}}^\vee_{B^\vee/C}, \mathscr{O}_C(1)), (L_0^\vee(1), L_{0,\mathsf{Z}}^\vee(1), R(1)) \underset{R}{\otimes} \mathscr{O}_C\big)$$

to the above $\mathcal{E}_{\mathrm{M}_0,C}$ induces a canonical isomorphism

$$\mathcal{E}_{\mathrm{P}_{0,\mathsf{Z}}} \overset{\mathrm{P}_{0,\mathsf{Z}}}{\times} \big(W|_{\mathrm{P}_{0,\mathsf{Z}}}\big) \overset{\sim}{\to} \mathcal{E}_{\mathrm{M}_0,C} \overset{\mathrm{M}_0}{\times} W = \mathcal{E}_C.$$

Let $\mathrm{U}_{0,\mathsf{Z}}$ denote the unipotent radical of $\mathrm{P}_{0,\mathsf{Z}}$, and let $\mathrm{M}_{0,\mathsf{Z}} := \mathrm{P}_{0,\mathsf{Z}}/\mathrm{U}_{0,\mathsf{Z}}$ denote its Levi quotient. Then, by the same arguments as in the proofs of [35, Lem. 1.18, Lem. 1.20, and Cor. 1.21], the filtration $\{W^i\}_{i\in\mathbb{Z}}$ on $W|_{\mathrm{P}_{0,\mathsf{Z}}}$ induced by the action of $\mathrm{U}_{0,\mathsf{Z}}$ (whose graded pieces are $\mathrm{M}_{0,\mathsf{Z}}$-modules) defines a filtration $\{\mathcal{E}_C^i\}_{i\in\mathbb{Z}}$ on $\mathcal{E}_C$ with graded pieces

$$\mathcal{E}_C^i/\mathcal{E}_C^{i+1} \cong (C \to \mathsf{Z})^*\Big(\mathcal{E}_{\mathrm{M}_{0,\mathsf{Z}}} \overset{\mathrm{M}_{0,\mathsf{Z}}}{\times} \big(W^i/W^{i+1}\big)\Big),$$

where

$$\mathcal{E}_{\mathrm{M}_{0,\mathsf{Z}}} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{Z}}}\big((\underline{\mathrm{Lie}}^\vee_{T^\vee/\mathsf{Z}}, \underline{\mathrm{Lie}}^\vee_{B^\vee/\mathsf{Z}}, \mathscr{O}_{\mathsf{Z}}(1)), (L_0^\vee(1)/L_{0,\mathsf{Z}}^\vee(1), L_{0,\mathsf{Z}}^\vee(1), R(1)) \underset{R}{\otimes} \mathscr{O}_{\mathsf{Z}}\big).$$

(The morphisms $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0} \to \mathsf{M}^{\mathrm{tor}}$, $\mathcal{E}_{\mathrm{M}_0,C} \to C$, $\mathcal{E}_{\mathrm{P}_{0,\mathrm{Z}}} \to C$, and $\mathcal{E}_{\mathrm{M}_{0,\mathrm{Z}}} \to \mathsf{Z}$ all have sections locally over their bases, because they are of finite presentation and have sections over the local rings of their bases, by comparing $\mathcal{O}$-multiranks as in [32, Lem. 1.1.3.1 and Cor. 1.1.2.6; cf. the proof of Lem. 1.1.3.4].) Thus the second assertion of the proposition has also been verified. □

Remark 5.7. The proof of Proposition 5.6 uses some special features of the PEL-type setup in [32] which are not necessarily shared by other setups. Nevertheless, for general complex-analytically constructed compactifications as in [3] etc, the statements of Proposition 5.6 are also satisfied by canonical extensions defined there, and they are easier to verify because the analytic construction uses coordinates more directly related to the algebraic groups. In fact, the statements can be viewed as characterizing properties for canonical extensions in both the analytic and algebraic setups. (See [21, (4.1.1) and (4.2.2)] for the analytic setup. See [30, Thm. 5.2.12 and Rem. 5.2.14] for a comparison between the two setups in PEL-type cases.)

Corollary 5.8. *The pullback $\mathscr{E}$ of $\mathcal{E}^{\mathrm{can}}$ to $\mathfrak{Y}$ (see Definition 4.11) is canonically isomorphic to the pullback of a canonically determined locally free coherent sheaf $\mathscr{E}_0$ over $C^{\wedge}_{\bar{x}}$ under the canonical morphism $\mathfrak{Y} \to C^{\wedge}_{\bar{x}}$. Moreover, $\mathscr{E}_0$ admits a filtration such that its graded pieces are isomorphic to pullbacks of coherent sheaves over $\mathsf{M}_0 = \mathrm{Spec}(R)$.*

By (5.5) and Corollary 5.8, and by the projection formula [18, 0_{I}, 5.4.10.1]:

Corollary 5.9. *With the sheaf $\mathscr{E}_0$ as in Corollary 5.8, the pullbacks of $\mathcal{E}^{\mathrm{can}}(n\mathsf{D}')$ and $\mathcal{E}^{\mathrm{sub}}(n\mathsf{D}')$ to $\mathfrak{V}_\sigma$ are respectively given by the $\mathscr{O}_{C^{\wedge}_{\bar{x}}}$-modules $\hat{\bigoplus}_{\ell \in \sigma^\vee_{(n)}} \Big((\Psi(\ell))^{\wedge}_{\bar{x}} \underset{\mathscr{O}_{C^{\wedge}_{\bar{x}}}}{\otimes} \mathscr{E}_0 \Big)$ and $\hat{\bigoplus}_{\ell \in \sigma^\vee_{(n)+}} \Big((\Psi(\ell))^{\wedge}_{\bar{x}} \underset{\mathscr{O}_{C^{\wedge}_{\bar{x}}}}{\otimes} \mathscr{E}_0 \Big)$.*

Definition 5.10. *Let $\mathscr{E}$ be as in Definition 4.11, and let $\mathscr{E}_0$ be as in Corollary 5.8. For each $\ell \in \mathbf{S}$ and each $d \in \mathbb{Z}$, we define the* **Fourier–Jacobi coefficient module** $\mathrm{FJ}^{d,(\ell)}(\mathscr{E}) := H^d(C^{\wedge}_{\bar{x}}, (\Psi(\ell))^{\wedge}_{\bar{x}} \underset{\mathscr{O}_{C^{\wedge}_{\bar{x}}}}{\otimes} \mathscr{E}_0)$.

Corollary 5.11. *For each $\sigma \in \Sigma^+$ and any integers $n' \geq n$, we have a commutative diagram*

$$\begin{array}{ccc} H^d(\mathfrak{V}_\sigma, \mathscr{E}^{(n)+}|_{\mathfrak{V}_\sigma}) & \longrightarrow & H^d(\mathfrak{V}_\sigma, \mathscr{E}^{(n)}|_{\mathfrak{V}_\sigma}) \\ \downarrow & & \downarrow \\ H^d(\mathfrak{V}_\sigma, \mathscr{E}^{(n')+}|_{\mathfrak{V}_\sigma}) & \longrightarrow & H^d(\mathfrak{V}_\sigma, \mathscr{E}^{(n')}|_{\mathfrak{V}_\sigma}) \end{array}$$

of canonical morphisms given by the commutative diagram

$$\begin{array}{ccc} \hat{\bigoplus}_{\ell \in \sigma^\vee_{(n)+}} \mathrm{FJ}^{d,(\ell)}(\mathscr{E}) & \longrightarrow & \hat{\bigoplus}_{\ell \in \sigma^\vee_{(n)}} \mathrm{FJ}^{d,(\ell)}(\mathscr{E}) \\ \downarrow & & \downarrow \\ \hat{\bigoplus}_{\ell \in \sigma^\vee_{(n')+}} \mathrm{FJ}^{d,(\ell)}(\mathscr{E}) & \longrightarrow & \hat{\bigoplus}_{\ell \in \sigma^\vee_{(n')}} \mathrm{FJ}^{d,(\ell)}(\mathscr{E}) \end{array}$$

of canonical morphisms.

Corollary 5.12. *For $? = \emptyset$ or $+$, and for $\sigma, \tau \in \Sigma^+$ such that τ is a face of σ, we have $\sigma^\vee_{(n)?} \subset \tau^\vee_{(n)?}$ by definition, and the canonical morphism*

$$H^d(\mathfrak{V}_\sigma, \mathscr{E}^{(n)?}|_{\mathfrak{V}_\sigma}) \to H^d(\mathfrak{V}_\tau, \mathscr{E}^{(n)?}|_{\mathfrak{V}_\tau})$$

is given by the canonical morphism

$$\hat{\bigoplus_{\ell \in \sigma^\vee_{(n)?}}} \mathrm{FJ}^{d,(\ell)}(\mathscr{E}) \to \hat{\bigoplus_{\ell \in \tau^\vee_{(n)?}}} \mathrm{FJ}^{d,(\ell)}(\mathscr{E})$$

inducing the identity morphism on $\mathrm{FJ}^{d,(\ell)}(\mathscr{E})$ when $\ell \in \sigma^\vee_{(n)?}$.

Corollary 5.13. *For $? = \emptyset$ or $+$, for each $\sigma \in \Sigma^+$, and for each $\gamma \in \Gamma$, we have $(\gamma\sigma)^\vee_{(n)?} = \gamma(\sigma^\vee_{(n)?})$, and the canonical isomorphism*

$$\gamma^* : H^d(\mathfrak{V}_{\gamma\sigma}, \mathscr{E}^{(n)?}|_{\mathfrak{V}_{\gamma\sigma}}) \to H^d(\mathfrak{V}_\sigma, \mathscr{E}^{(n)?}|_{\mathfrak{V}_\sigma})$$

is given by the canonical morphism

$$\hat{\bigoplus_{\ell \in \sigma^\vee_{(n)?}}} \mathrm{FJ}^{d,(\gamma\ell)}(\mathscr{E}) \to \hat{\bigoplus_{\ell \in \sigma^\vee_{(n)?}}} \mathrm{FJ}^{d,(\ell)}(\mathscr{E})$$

inducing, for each $\ell \in \sigma^\vee_{(n)?}$, the canonical isomorphism

$$\gamma^* : \mathrm{FJ}^{d,(\gamma\ell)}(\mathscr{E}) \xrightarrow{\sim} \mathrm{FJ}^{d,(\ell)}(\mathscr{E}) \tag{5.14}$$

given by the composition

$$\begin{aligned} H^d(C^\wedge_{\bar{x}}, (\Psi(\gamma\ell))^\wedge_{\bar{x}} \underset{\mathscr{O}_{C^\wedge_{\bar{x}}}}{\otimes} \mathscr{E}_0) &\xrightarrow{\sim} H^d(C^\wedge_{\bar{x}}, \gamma^*(\Psi(\gamma\ell))^\wedge_{\bar{x}} \underset{\mathscr{O}_{C^\wedge_{\bar{x}}}}{\otimes} \mathscr{E}_0) \\ &\xrightarrow{\sim} H^d(C^\wedge_{\bar{x}}, (\Psi(\ell))^\wedge_{\bar{x}} \underset{\mathscr{O}_{C^\wedge_{\bar{x}}}}{\otimes} \mathscr{E}_0), \end{aligned}$$

where the first isomorphism is induced by the automorphism $\gamma : C^\wedge_{\bar{x}} \xrightarrow{\sim} C^\wedge_{\bar{x}}$ over $\mathsf{Z}^\wedge_{\bar{x}}$, and where the second isomorphism is induced by (4.10).

6. Positivity

The goal of this section is to prove Theorem 2.3 using Lemma 4.14.

First let us introduce a canonical factorization of $\mathbf{S}_\mathbb{Q} := \mathbf{S} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ into a product of $\mathbb{Q}$-vector spaces, whose factors will be called its $\mathbb{Q}$-*simple factors*. In the setup of [32], the semisimple algebra $\mathcal{O}_\mathbb{Q} := \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ over $\mathbb{Q}$ (where $\mathcal{O}$ is part of the integral PEL datum $(\mathcal{O}, \star, L, \langle \cdot, \cdot \rangle, h_0)$ defining M) decomposes into a product of its $\mathbb{Q}$-simple factors (as in [32, (1.2.1.10)]), and $\mathbf{S}_\mathbb{Q}$ can be identified with the $\mathbb{Q}$-vector space of Hermitian pairings over some $\mathcal{O}_\mathbb{Q}$-module (cf. the beginning of [32, Sec. 6.2.5]); hence $\mathbf{S}_\mathbb{Q}$ factorizes as $\mathcal{O}_\mathbb{Q}$ does. In the setup of complex-analytically constructed compactifications as in [3] etc, $\mathbf{S}_\mathbb{Q}$ is canonically dual to the $\mathbb{Q}$-valued points of the center of the unipotent radical of some rational parabolic subgroup (associated with Z) of the reductive algebraic group G over $\mathbb{Q}$ defining the Shimura variety in question; hence $\mathbf{S}_\mathbb{Q}$ factorizes as G^{ad} (as an algebraic group over $\mathbb{Q}$) does. (See [30, Cor. 3.6.10] for a comparison between the two setups in PEL-type cases.)

Let $\mathbf{S}_\mathbb{Q} = (\mathbf{S}_\mathbb{Q})_0 \times (\mathbf{S}_\mathbb{Q})_1$ denote the factorization of $\mathbf{S}_\mathbb{Q}$ such that $(\mathbf{S}_\mathbb{Q})_1$ is the product of all $\mathbb{Q}$-simple factors of $\mathbf{S}_\mathbb{Q}$ (in the above sense) isomorphic to $\mathbb{Q}$. Accordingly, we have the induced factorizations $\mathbf{S}^\vee_\mathbb{R} = (\mathbf{S}^\vee_\mathbb{R})_0 \times (\mathbf{S}^\vee_\mathbb{R})_1$, $\mathbf{P} = \mathbf{P}_0 \times \mathbf{P}_1$, and $\mathbf{P}^+ = \mathbf{P}^+_0 \times \mathbf{P}^+_1$. Let pr_0 and pr_1 denote the canonical projections to the first and second factors, respectively, in any of these factorizations. Note that an element

$\ell_1 \in (\mathbf{S}_{\mathbb{Q}})_1$ lies in $\mathbf{P}_1^\vee$ if, in the factorization of $(\mathbf{S}_{\mathbb{Q}})_1$ as products of copies of $\mathbb{Q}$, none of the components of ℓ_1 is negative.

Let us denote by $\mathbf{P}^{\vee,+}$ the subset of $\mathbf{P}^\vee$ consisting of elements $\ell \in \mathbf{S}$ that pair positively with all nonzero elements in $\mathbf{P}$.

Lemma 6.1. *If $\ell \in \mathbf{P}^{\vee,+}$, then $\mathrm{FJ}^{d,(\ell)}(\mathscr{E}) = 0$ for all $d > 0$.*

Proof. By considering the spectral sequence associated with the filtration in the second assertion of Corollary 5.8, the question is reduced to showing that, for every finite R-module M, we have $H^d(C^\wedge_{\bar{x}}, (\Psi(\ell))^\wedge_{\bar{x}} \underset{R}{\otimes} M) = 0$ for all $d > 0$. By the same reduction argument as in the proof of [32, Lem. 7.1.1.4] (viewing M as an R_0-module), we may assume that $R = M$ is a quotient ring of R_0, and the question is reduced to showing that $H^d(C^\wedge_{\bar{x}}, (\Psi(\ell))^\wedge_{\bar{x}}) = 0$ for all $d > 0$. Since $\ell \in \mathbf{P}^{\vee,+}$, the invertible sheaf $\Psi(\ell)$ over $C \to \mathsf{Z}$ is relatively ample. (In the setup of [32], the proof is as in [32, the fifth paragraph of the proof of Thm. 7.3.3.4(1), p. 482] and [31, Cor. 2.12 and Lem. 5.5]. In the setup of complex-analytically constructed compactifications as in [3] etc, this follows from the analysis of Siegel domains of the third kind, as explained in [42, Ch. 5, Sec. 1–2] and [5, Sec. 3], and from the classification of ample line bundles on complex abelian varieties as in [38, Sec. 2–3].) Hence we have the desired vanishing $H^d(C^\wedge_{\bar{x}}, (\Psi(\ell))^\wedge_{\bar{x}}) = 0$ for all $d > 0$ (see [38, Sec. 16] for the vanishing over the fiber over $\bar{x}$; and see [38, Sec. 5] and [18, III, 7.7.5 and 7.7.10] for the base change argument). □

Lemma 6.2. *Suppose $\ell \in \mathbf{S}$ is an element (also considered as an element of $\mathbf{S}_{\mathbb{Q}}$) such that $\mathrm{pr}_1(\ell) \notin \mathbf{P}_1^\vee$. Then $\mathrm{FJ}^{d,(\ell)}(\mathscr{E}) = 0$ for all $d < c_{\mathsf{M}} - 1$.*

Proof. By the same argument as in the proof of Lemma 6.1, the question is reduced to showing that $H^d(C^\wedge_{\bar{x}}, (\Psi(\ell))^\wedge_{\bar{x}}) = 0$ for all $d < c_{\mathsf{M}} - 1$. Suppose $(\mathbf{S}_{\mathbb{Q}})_1 \cong \prod_v \mathbb{Q}_v$, where each $\mathbb{Q}_v$ denotes a $\mathbb{Q}$-simple factor $\mathbb{Q}$ of $(\mathbf{S}_{\mathbb{Q}})_1$. By construction of C (see [32, Sec. 6.2]; cf. [4, Ch. III, Sec. 4–6]), there exists an isogeny to C from a fiber product $C_0 \times \prod_v C_v$ of abelian schemes over a finite étale extension over Z, where each C_v corresponds to the $\mathbb{Q}_v$ above, such that the pullback of $\Psi(\ell)$ is isomorphic to an exterior product $\Psi_0(\ell_0) \boxtimes (\boxtimes_v \Psi_v(\ell_v))$, where $\Psi_0(\ell_0)$ is an invertible sheaf over C_0 and where $\Psi_v(\ell_v)$ is an invertible sheaf over C_v for each v; moreover, $\Psi_v(\ell_v)$ is relatively ample over $C_v \to \mathsf{Z}$ if and only if ℓ_v is positive in $\mathbb{Q}_v$ (cf. the references in the proof of Lemma 6.1). If $\mathrm{pr}_1(\ell) \notin \mathbf{P}_1^\vee$, which means $\ell_v < 0$ for some v, then $\Psi_v(-\ell_v) \cong \Psi_v(\ell_v)^{\otimes(-1)}$ is relatively ample over $(C_v)^\wedge_{\bar{x}} \to \mathsf{Z}^\wedge_{\bar{x}}$, and hence $H^a((C_v)^\wedge_{\bar{x}}, (\Psi(\ell))^\wedge_{\bar{x}}) = 0$ for all $a < d_v := \dim(C_v) - \dim(\mathsf{Z})$ (see [38, Sec. 16] for the vanishing over the fiber over $\bar{x}$; and see [38, Sec. 5] and [18, III, 7.7.5 and 7.7.10] for the base change argument). Consequently, by the Künneth formula [18, III-2, 6.7.8], $H^d(C^\wedge_{\bar{x}}, (\Psi(\ell))^\wedge_{\bar{x}}) = 0$ for all $d < d_v$. By construction (see [32, Sec. 6.2]; cf. [4, Ch. III, Sec. 4–6]), for any maximal strata Z' of $\mathsf{M}^{\min} - \mathsf{M}$ sharing the factors $\mathbb{Q}_v$ and C_v, and containing Z in its closure, we have $d_v = \mathrm{codim}(\mathsf{Z}', \mathsf{M}^{\min}) - 1$. Consequently, $H^d(C^\wedge_{\bar{x}}, (\Psi(\ell))^\wedge_{\bar{x}}) = 0$ for all $d < c_{\mathsf{M}} - 1 \leq d_v$, as desired. □

Lemma 6.3. *For each integer $n \geq 0$, if $\ell \in \mathbf{S}$, if $\mathrm{pr}_1(\ell) \in \mathbf{P}_1^\vee$, and if $\ell \in \sigma^\vee_{(n)}$ for all $\sigma \in \Sigma^+$, then $\ell \in \mathbf{P}^\vee$.*

Proof. Since pol is Γ-invariant and $\underset{\sigma \in \Sigma^+}{\cup} \overline{\sigma} = \mathbf{P}$, the condition that $\ell \in \sigma^\vee_{(n)}$ for all $\sigma \in \Sigma^+$ implies that $\langle \ell, \gamma y \rangle \geq -n\mathsf{pol}(y)$ for all $\gamma \in \Gamma$ and $y \in \mathbf{P}^+$.

Suppose $\ell \notin \mathbf{P}^\vee$. By definition, there exists an element $y_0 \in \mathbf{P}$ such that $\langle \ell, y_0 \rangle < 0$. Since $\mathbf{P}^+$ is dense in $\mathbf{P}$, we may and we shall assume that $y_0 \in \mathbf{P}^+$. By the first paragraph above, in order to prove the lemma, it suffices to show that there exists a sequence of elements $\{\gamma_j\}_{j \in \mathbb{Z}_{>0}}$ such that

$$\lim_{j \to +\infty} \langle \ell, \gamma_j y_0 \rangle \to -\infty. \tag{6.4}$$

In each case, Γ is an arithmetic subgroup (of finite covolume) of a real reductive group H acting on $\mathbf{S}_\mathbb{R}$ and preserving $\mathbf{P}^+$, or more precisely preserving any characteristic function $\varphi : \mathbf{P}^+ \to \mathbb{R}_{>0}$ as in [4, Ch. II, Sec. 1.2]. By a result of Selberg's (see [7, Lem. 1.4]), the action of H can be approximated by that of Γ (by pre- and post- compositions with elements in any prescribed neighborhood of the identity). Hence it suffices to find elements $\{h_j\}_{j \in \mathbb{Z}_{>0}}$ in H such that

$$\lim_{j \to +\infty} \langle \ell, h_j y_0 \rangle \to -\infty. \tag{6.5}$$

In both [5, p. 299] and [42, Ch. 4, Sec. 1, Lem. 2], it was asserted that (6.5) can be shown by easy case-by-case arguments. Let us give a uniform argument: In each case, $\mathbf{S}_\mathbb{R}$ admits the structure of a formally real Jordan algebra whose trace pairing induces via $\langle \cdot, \cdot \rangle$ an isomorphism $\mathbf{S}_\mathbb{R} \cong \mathbf{S}_\mathbb{R}^\vee$ such that the nonnegative elements in $\mathbf{S}_\mathbb{R}$ form the closure $\overline{\mathbf{P}^\vee}$ of $\mathbf{P}^\vee$ in $\mathbf{S}_\mathbb{R}$. The formally real Jordan algebra $\mathbf{S}_\mathbb{R}$ factorizes into a product of its $\mathbb{R}$-simple factors, each factor being contained in the base change to $\mathbb{R}$ of some $\mathbb{Q}$-simple factor of $\mathbf{S}_\mathbb{Q}$ introduced above. We shall call these base changes the $\mathbb{Q}$-simple factors of $\mathbf{S}_\mathbb{R}$. By the spectral theorem for formally real Jordan algebras (see, for example, [11, Ch. III]), for each $x \in \mathbf{S}_\mathbb{R}$, there exist mutually orthogonal idempotents $\{e_i\}_i$ in $\mathbf{S}_\mathbb{R}$ summing up to 1, and real numbers $\{c_i\}_i$, both sets depending on x, such that $x = \sum_i c_i e_i$; and $x \in \overline{\mathbf{P}^\vee}$ if and only if all $c_i \geq 0$. Moreover, for each index i, the corresponding e_i belongs to some $\mathbb{R}$-simple factor of $\mathbf{S}_\mathbb{R}$, and the induced action of H preserves the product $\prod_j c_j$ over all the indices j such that e_i and e_j belong to the same $\mathbb{Q}$-simple factor of $\mathbf{S}_\mathbb{R}$. Since $\ell \notin \mathbf{P}^\vee = \overline{\mathbf{P}^\vee} \cap \mathbf{S}$ (and so $\ell \notin \overline{\mathbf{P}^\vee}$ because $\ell \in \mathbf{S}$) and since $\langle \ell, y_0 \rangle < 0$, we can write $\ell = \sum_i c_i e_i$ as above, with $c_{i_0} < 0$ and $\langle e_{i_0}, y_0 \rangle > 0$ for some i_0. Since $\mathrm{pr}_1(\ell) \in \mathbf{P}_1^\vee$, the $\mathbb{Q}$-simple factor of $\mathbf{S}_\mathbb{R}$ containing e_{i_0} cannot be isomorphic to $\mathbb{R}$. Hence we can achieve (6.5) by choosing h_j such that $h_j y_0 = \sum_i d_{j,i} e_i$, where e_i are the same ones determined by ℓ above, and where $d_{j,i}$ are positive real numbers such that $d_{j,i_0} \to +\infty$ and $\{d_{j,i}\}_{i \neq i_0}$ stay bounded as $j \to +\infty$. □

Remark 6.6. For H as in the proof of Lemma 6.3, the $\mathbb{R}$-simple factors of $\mathrm{Lie}(\mathrm{H})^{\mathrm{ad}}$ are isomorphic to either $\mathfrak{sl}_{r,\mathbb{R}}$, $\mathfrak{sl}_{r,\mathbb{C}}$, $\mathfrak{sl}_{r,\mathbb{H}}$ $(= \mathfrak{su}_{2r}^*)$, $\mathfrak{so}_{1,r-1}$, for some $r > 1$, or $\mathfrak{e}_{6(-26)}$ (cf. [25, Ch. X, Sec. 6, Table V]). Accordingly, if H_0 is any maximal compact subgroup of H (which is the stabilizer of some half-line in $\mathbf{P}^+$), then the $\mathbb{R}$-simple factors of $\mathrm{Lie}(\mathrm{H}_0)^{\mathrm{ad}}$ are isomorphic to either $\mathfrak{so}_r$, $\mathfrak{su}_r$, $\mathfrak{su}_{r,\mathbb{H}}$ $(= \mathfrak{sp}_r)$, $\mathfrak{so}_{r-1}$, or $\mathfrak{f}_4$. Only the first three cases can occur in PEL-type cases; the case (of $\mathrm{Lie}(\mathrm{H})^{\mathrm{ad}}$) with $\mathfrak{e}_{6(-26)}$ factors can occur only in cases (of models of Shimura varieties, or disjoint unions of their connected components) with type-E_7 factors. The corresponding $\mathbb{R}$-simple factors of $\mathbf{S}_\mathbb{R}$ are isomorphic to the spaces of $r \times r$ Hermitian matrices over either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (the Hamiltonian numbers); or to $\mathbb{R}^r$, for some $r > 1$; or to the space of 3×3 Hermitian matrices over $\mathbb{O}$ (the Cayley numbers, an octonion normed division algebra over $\mathbb{R}$). The corresponding $\mathbb{R}$-simple factors of $\mathbf{P}^+$ are isomorphic to the subsets of positive definite matrices in all but the second last case, where $\mathbf{P}^+$ is isomorphic to the light cone $x_1^2 > x_2^2 + \cdots x_r^2$. (See, for example, [4, Ch. II, Rem. 1.11] and [11, Ch. V, Sec. 3].) (In the symplectic and unitary PEL-type cases, the spectral theorem in the proof of Lemma 6.3 is nothing but the familiar one over $\mathbb{R}$ and $\mathbb{C}$.)

Proposition 6.7. *For all integers $n \geq 0$, and for all $d < c_{\mathsf{M}} - 1$, we have canonical Γ-equivariant isomorphisms*

$$H^0(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^d(\mathscr{E})) \xrightarrow{\sim} H^0(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^d(\mathscr{E}^{(n)})) \cong \hat{\bigoplus_{\ell \in \mathbf{P}^\vee}} \mathrm{FJ}^{d,(\ell)}(\mathscr{E}), \tag{6.8}$$

where the Γ-action on $\hat{\bigoplus}_{\ell \in \mathbf{P}^\vee} \mathrm{FJ}^{d,(\ell)}(\mathscr{E})$ is induced by the isomorphisms (5.14).

Proof. This follows from Corollaries 5.9, 5.11, 5.12, and 5.13, and from Lemmas 6.2 and 6.3. □

When $c_{\mathsf{M}} > 1$, it follows from Proposition 6.7 (with $d = 0$) that (4.15) is an isomorphism. Thus we have proved Theorem 2.3, by Lemma 4.14.

7. Vanishing of higher direct images

The goal of this section is to prove Theorem 3.9 using Lemma 4.16.

Lemma 7.1. *Let $\sigma = \mathbb{R}_{>0}v_1 + \cdots + \mathbb{R}_{>0}v_r$ be a nonzero smooth nondegenerate rational polyhedral cone in $\mathbf{S}_{\mathbb{R}}^\vee$, where $v_1, \ldots, v_r$ are nonzero vectors, and let $\mathbf{K}$ be a subset of $\mathbf{S}_{\mathbb{R}}^\vee$ stable under the action of $\mathbb{R}_{>0}^\times$ such that $0 \notin \mathbf{K}$ and $\overline{\sigma}_{\mathbf{K}} := \overline{\sigma} \cap \mathbf{K}$ is convex. (Here $\overline{\sigma}$ is the closure of σ in $\mathbf{S}_{\mathbb{R}}^\vee$.) Up to reordering $v_1, \ldots, v_r$ if necessary, suppose moreover that, for some $0 < m \leq r$, we have*

$$(\mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_m) - \{0\} \subset \mathbf{K} \tag{7.2}$$

but

$$(\mathbb{R}_{\geq 0}v_{m+1} + \cdots + \mathbb{R}_{\geq 0}v_r) \cap \mathbf{K} = \emptyset. \tag{7.3}$$

In this case, by smoothness of σ, the cone $\tau := \mathbb{R}_{>0}v_1 + \ldots + \mathbb{R}_{>0}v_m$ is the largest face of σ such that its closure $\overline{\tau}$ (in $\mathbf{S}_{\mathbb{R}}^\vee$) satisfies $\overline{\tau} - \{0\} \subset \mathbf{K}$ (so that $\overline{\tau} - \{0\} \subset \overline{\sigma}_{\mathbf{K}}$). Consider the continuous map

$$F : [0,1] \times \overline{\sigma}_{\mathbf{K}} \to \overline{\sigma}_{\mathbf{K}}$$

defined by sending

$$(t, x_1v_1 + \cdots + x_mv_m + x_{m+1}v_{m+1} + \cdots + x_rv_r)$$

to

$$x_1v_1 + \cdots + x_mv_m + (1-t)x_{m+1}v_{m+1} + \cdots + (1-t)x_rv_r.$$

Then F defines a deformation retract from $\overline{\sigma}_{\mathbf{K}}$ to its subset $\overline{\tau} - \{0\}$. The construction of F is compatible with restrictions to faces ρ of σ that still satisfy the condition of this lemma.

Proof. The statements are self-explanatory. (The condition (7.2) is needed for the compatibility with restrictions to faces. The condition (7.3) is needed for the deformation retract F to be defined—i.e., nonzero—at $t = 1$.) □

Definition 7.4. *For each $\ell \in \mathbf{S}$ and each $n \in \mathbb{Z}$, and for $? = \emptyset$ or $+$, we define $\widetilde{\mathfrak{N}}^{\ell,(n)?}$ to be the subset of $\widetilde{\mathfrak{N}}$ formed by the union of $\sigma \in \Sigma^+$ such that $\ell \in \sigma^\vee_{(n)?}$. (See* (5.3) *and Lemma* 5.4.*)*

For simplicity, let $\mathbf{P}' := \mathbf{P} - \{0\}$. Lemma 7.5 below also works if we replace $\mathbf{P}'$ with any convex subset of $\mathbf{P} - \{0\}$ formed by unions of cones in Σ, and accordingly replace $\widetilde{\mathfrak{N}}$, $\widetilde{\mathfrak{N}}^{\ell,(-n)}$, and $\widetilde{\mathfrak{N}}^{\ell,(-n)+}$ with their intersections with $\mathbf{P}'$.

Lemma 7.5. *For each $\ell \in \mathbf{S}$ and all integers $n \geq 0$, the sets $\widetilde{\mathfrak{N}} - \widetilde{\mathfrak{N}}^{\ell,(-n)}$ and $\widetilde{\mathfrak{N}} - \widetilde{\mathfrak{N}}^{\ell,(-n)+}$ are either contractible or empty.*

Proof. Let
$$\mathbf{P}'_{\ell<(-n)} := \{y \in \mathbf{P}' : \langle \ell, y\rangle < n\mathsf{pol}(y)\}$$
and
$$\mathbf{P}'_{\ell\leq(-n)} := \{y \in \mathbf{P}' : \langle \ell, y\rangle \leq n\mathsf{pol}(y)\}.$$
Since pol satisfies $\mathsf{pol}(y+z) \geq \mathsf{pol}(y) + \mathsf{pol}(z)$ for all $y, z \in \mathbf{P}$, and since $n \geq 0$, these are convex subsets of $\mathbf{P}'$. Consider the canonical embeddings
$$\widetilde{\mathfrak{N}} \cap \mathbf{P}'_{\ell<(-n)} \hookrightarrow \widetilde{\mathfrak{N}} - \widetilde{\mathfrak{N}}^{\ell,(-n)} \tag{7.6}$$
and
$$\widetilde{\mathfrak{N}} \cap \mathbf{P}'_{\ell\leq(-n)} \hookrightarrow \widetilde{\mathfrak{N}} - \widetilde{\mathfrak{N}}^{\ell,(-n)+}. \tag{7.7}$$
Consider any $\sigma \in \Sigma^+$ such that $\overline{\sigma} - \{0\}$ has a nonempty intersection with $\widetilde{\mathfrak{N}} - \widetilde{\mathfrak{N}}^{\ell,(-n)}$ (resp. $\widetilde{\mathfrak{N}} - \widetilde{\mathfrak{N}}^{\ell,(-n)+}$). Up to replacing the cone decomposition with some smooth locally finite refinement without changing pol (see Remark 4.17) and without changing the two sides of (7.6) (resp. (7.7)), we may assume that, for each σ as above, there exists at least one face τ of σ such that $\overline{\tau} - \{0\}$ is contained in $\widetilde{\mathfrak{N}} \cap \mathbf{P}'_{\ell<(-n)}$ (resp. $\widetilde{\mathfrak{N}} \cap \mathbf{P}'_{\ell\leq(-n)}$). (Note that the restriction of pol to each such σ is a linear function.) Since $\widetilde{\mathfrak{N}} = \mathbf{P}^+$ and $\mathbf{P}'_{\ell<(-n)}$ (resp. $\mathbf{P}'_{\ell\leq(-n)}$) are convex subsets of $\mathbf{P}'$, both being stable under the action of $\mathbb{R}^\times_{>0}$, by Lemma 7.1, there are deformation retracts, compatible with restrictions to faces, from both $\overline{\sigma} - \overline{\rho}$ and $(\overline{\sigma} - \{0\}) \cap \mathbf{P}'_{\ell<(-n)}$ (resp. $(\overline{\sigma} - \{0\}) \cap \mathbf{P}'_{\ell\leq(-n)}$) to $\overline{\tau} - \{0\}$, where τ is the largest face of σ such that $\overline{\tau} - \{0\}$ is contained in $\mathbf{P}'_{\ell<(-n)}$ (resp. $\mathbf{P}'_{\ell\leq(-n)}$), and where ρ is the largest face of σ such that $\overline{\tau} - \{0\} \subset \overline{\sigma} - \overline{\rho}$. (Such τ and ρ uniquely exist because σ is smooth.) Hence we see that (7.6) (resp. (7.7)) is a homotopy equivalence. Since convex subsets of $\mathbf{P}'$ are either contractible or empty, the lemma follows. □

Proposition 7.8. *For all integers $n \geq 0$, $d \geq 0$, and $e > 0$, we have*
$$H^e(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^d(\mathscr{E}^{(-n)+})) \xrightarrow{\sim} H^e(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^d(\mathscr{E}^{(-n)})) = 0. \tag{7.9}$$

Proof. By Lemma 7.5, for each $\ell \in \mathbf{S}$, the local cohomology of $\widetilde{\mathfrak{N}}$ supported on $\widetilde{\mathfrak{N}} - \widetilde{\mathfrak{N}}^{\ell,(-n)}$ (resp. $\widetilde{\mathfrak{N}} - \widetilde{\mathfrak{N}}^{\ell,(-n)+}$) vanishes in all degrees $e > 0$. Hence, by Corollaries 5.9 and 5.12, we have (7.9) for $e > 0$ by the usual weight-by-weight argument, as in [26, Ch. I, Sec. 3]. (Since the nerves involve infinitely many cones, let us briefly review why one can still work weight by weight. This is because, up to replacing the cone decomposition Σ with locally finite refinements not necessarily carrying Γ-actions, which is harmless for proving this proposition, we can compute the cohomology as a limit using unions of finite cone decompositions on expanding convex polyhedral subcones, by proving inductively that the cohomology of one degree lower has the desired properties, using [50, Thm. 3.5.8]; then we can consider the associated graded pieces defined by the completions, and work weight-by-weight with subsheaves of $\underline{\mathscr{H}}^d(\mathscr{E}^{(-n)})(\sigma^{\mathrm{cl}})$ of the form $\mathrm{FJ}^{d,(\ell)}(\mathscr{E})$, because taking cohomology commutes with taking infinite direct sums for Čech complexes defined by finite coverings.) □

Remark 7.10. The arguments in the proof of Proposition 7.8 and in the preparation leading up to it have also appeared in slightly different forms in [31, Sec. 4], [22], and [28, Sec. 7.3 and Sec. 8.2], which can be traced back to much earlier sources in the literature: The consideration of contractibility can be found in [26, Ch. I, Sec. 3] and [23] (cf. Remark 3.10), and the consideration of expanding convex polyhedral subcones can be found in [10, Ch. VI, Sec. 2].

Thus we have verified (1) of Lemma 4.16.

Lemma 7.11. *For $\sigma \in \Sigma^+$ and for all integers $n' > n \geq 0$, we have $\sigma^\vee_{(-n')+} \subset \sigma^\vee_{(-n')} \subset \sigma^\vee_{(-n)+}$.*

Proof. This follows from Lemma 5.4, because $\mathsf{pol}(\mathbf{P} - \{0\}) \subset \mathbb{R}_{>0}$ by definition (see [32, Def. 7.3.1.1]). □

Corollary 7.12. *Given integers $n' > n \geq 0$, for $? = (-n')$ or $(-n)+$, let*

$$\mathbf{P}^{\vee,?} := \bigcap_{\sigma \in \Sigma^+} \sigma^\vee_?. = \{\ell \in \mathbf{S} : \ell \in \sigma^\vee_? \text{ for all } \sigma \in \Sigma^+\}.$$

Then $\mathbf{P}^{\vee,(-n')} \subset \mathbf{P}^{\vee,(-n)+} \subset \mathbf{P}^{\vee,+} = \bigcap_{\sigma \in \Sigma^+} \sigma^\vee_{(0)+}$ are Γ-stable subsets.

Proof. This follows from Lemma 7.11 and from the definitions. □

Proposition 7.13. *Given any integers $n' > n \geq 0$, for $? = (-n')$ or $(-n)+$, we have a canonical Γ-equivariant isomorphism*

$$H^0(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^d(\mathscr{E}^?)) \cong \hat{\bigoplus_{\ell \in \mathbf{P}^{\vee,?}}} \mathrm{FJ}^{d,(\ell)}(\mathscr{E}), \tag{7.14}$$

where the Γ-action on $\hat{\bigoplus}_{\ell \in \mathbf{P}^{\vee,?}} \mathrm{FJ}^{d,(\ell)}(\mathscr{E})$ is induced by the isomorphisms (5.14).

Proof. This follows from Corollaries 5.11, 5.12, and 5.13, and 7.12. □

Lemma 7.15. *Suppose $\ell_0 \in \mathbf{P}^{\vee,+}$. Under the running assumption that the level $\mathcal{H}$ is neat, so that Γ is also neat, the stabilizer Γ_{ℓ_0} of ℓ_0 in Γ is trivial.*

Proof. In all cases (including non-PEL-type ones), by the general theory of self-adjoint homogeneous cones (see, for example, [11, Ch. I, Sec. 4]), Γ_{ℓ_0} can be identified with a discrete subgroup of a compact real reductive group stabilizing some point of a Riemannian symmetric space (cf. Remark 6.6); hence Γ_{ℓ_0} is necessarily finite. Consequently, the eigenvalues of elements in Γ_{ℓ_0} under any faithful representation are roots of unity, which must be equal to 1 because Γ is neat. Thus the finite group Γ_{ℓ_0} is trivial for all $\ell_0 \in \mathbf{P}^{\vee,+}$, as desired. □

Proposition 7.16. *With the setting as in Proposition 7.13, for $? = (-n')$ or $(-n)+$, we have*

$$H^c(\Gamma, H^0(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^d(\mathscr{E}^?))) \overset{(7.14)}{\cong} H^c\Big(\Gamma, \hat{\bigoplus_{\ell \in \mathbf{P}^{\vee,?}}} \mathrm{FJ}^{d,(\ell)}(\mathscr{E})\Big) = 0 \tag{7.17}$$

when either $c > 0$ or $d > 0$.

Proof. By Lemma 6.1 and Corollary 7.12, $\mathrm{FJ}^{d,(\ell)}(\mathscr{E}) = 0$ for all $d > 0$ and $\ell \in \mathbf{P}^{\vee,?} \subset \mathbf{P}^{\vee,+}$. Hence (7.17) holds when $d > 0$. On the other hand, suppose $c > 0$. By Corollary 5.13, $H^c\Big(\Gamma, \hat{\bigoplus}_{\ell \in \mathbf{P}^{\vee,?}} \mathrm{FJ}^{d,(\ell)}(\mathscr{E})\Big)$ admits a filtration with graded pieces given by subquotients of

$$H^c\Big(\Gamma, \prod_{\ell \in \Gamma \cdot \ell_0} \mathrm{FJ}^{d,(\ell)}(\mathscr{E})\Big), \tag{7.18}$$

for ℓ_0 running through representatives of Γ-orbits in $\mathbf{P}^{\vee,?}$. By Lemma 7.15,

$$\prod_{\ell \in \Gamma \cdot \ell_0} \mathrm{FJ}^{d,(\ell)}(\mathscr{E}) \cong \mathrm{Coind}_{\Gamma_{\ell_0}}^{\Gamma}\big(\mathrm{FJ}^{d,(\ell_0)}(\mathscr{E})\big) = \mathrm{Coind}_{\{\mathrm{Id}\}}^{\Gamma}\big(\mathrm{FJ}^{d,(\ell_0)}(\mathscr{E})\big)$$

(see [9, Ch. III, Sec. 5]). Hence (7.18) is zero for all $c > 0$, by Shapiro's lemma (see [9, Ch. III, (6.2)]), and (7.17) also holds when $c > 0$, as desired. □

Remark 7.19. The arguments in the proof of Proposition 7.16, especially the use of Shapiro's lemma, have also appeared in [22] and [28, Sec. 8.2].

Thus we have also verified (2) of Lemma 4.16, and proved Theorem 3.9.

8. Argument by duality

The goal of this section is to deduce Theorem 2.5 from Theorem 3.9, or rather a weaker form of the latter (see Remark 8.11 below).

Theorem 8.1. *For all integers $n' \geq n \geq 0$, and for all $a < c_{\mathsf{M}} - 1$ (resp. $a = c_{\mathsf{M}} - 1$), the canonical morphism*

$$H^a(\mathsf{M}^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}(n\mathsf{D}')) \to H^a(\mathsf{M}^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}(n'\mathsf{D}')) \tag{8.2}$$

is bijective (resp. injective); or, equivalently, for all $a < c_{\mathsf{M}} - 1$, we have

$$H^a(\mathsf{M}^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}(n'\mathsf{D}')/\mathcal{E}^{\mathrm{can}}(n\mathsf{D}')) = 0. \tag{8.3}$$

Consider the ample invertible sheaf $\omega^{\min}$ over $\mathsf{M}^{\min}$, and consider its pullbacks $\omega \cong (j^{\min})^*\omega^{\min}$ over M and $\omega^{\mathrm{tor}} \cong \oint^* \omega^{\min}$ over $\mathsf{M}^{\mathrm{tor}}$, as in [32, Thm. 7.2.4.1 (2) and (3)]. (These are the *Hodge line bundles* mentioned in Section 1.) By construction, there exists a rank-one free R-module $W_\omega \in \mathrm{Rep}_R(\mathrm{M}_0)$ such that, for each $N \in \mathbb{Z}$, the automorphic bundle $\mathcal{E}_N := \mathcal{E}_{\mathrm{M}_0,R}(W \underset{R}{\otimes} W_\omega^{\otimes N})$ and its canonical extension $\mathcal{E}_N^{\mathrm{can}}$ satisfy $\mathcal{E}_N \cong \mathcal{E} \underset{\mathscr{O}_{\mathsf{M}}}{\otimes} \omega^{\otimes N}$ and $\mathcal{E}_N^{\mathrm{can}} \cong \mathcal{E}^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}}{\otimes} (\omega^{\mathrm{tor}})^{\otimes N}$.

Lemma 8.4. *Theorem 2.5 follows from Theorem 8.1.*

Proof. As explained in Section 3, we may assume that R is noetherian and replace S_0 etc with their base changes to R, and it suffices to show that (3.4) holds for all $n' \geq n \geq 0$ and all $a < c_{\mathsf{M}} - 1$. Since $\omega^{\min}$ is an invertible sheaf over $\mathsf{M}^{\min}$, by the projection formula [18, 0_{I}, 5.4.10.1], we have

$$\begin{aligned} &R^a \oint_*(\mathcal{E}_N^{\mathrm{can}}(n'\mathsf{D}')/\mathcal{E}_N^{\mathrm{can}}(n\mathsf{D}')) \\ &\cong R^a \oint_*(\mathcal{E}^{\mathrm{can}}(n'\mathsf{D}')/\mathcal{E}^{\mathrm{can}}(n\mathsf{D}')) \underset{\mathscr{O}_{\mathsf{M}^{\min}}}{\otimes} (\omega^{\min})^{\otimes N} \end{aligned} \tag{8.5}$$

for all a and N; and it suffices to show that, for some integer N, we have

$$R^a \oint_*(\mathcal{E}_N^{\mathrm{can}}(n'\mathsf{D}')/\mathcal{E}_N^{\mathrm{can}}(n\mathsf{D}')) = 0$$

for all $a < c_{\mathsf{M}} - 1$. Since $\omega^{\min}$ is ample over $\mathsf{M}^{\min}$, and since $\mathsf{M}^{\min}$ is projective over the noetherian base scheme $\mathsf{M}_0 = \operatorname{Spec}(R)$, by (8.5) and by Serre's fundamental theorem [18, III, 2.2.1], there exists some integer N_0 such that, for every $N \geq N_0$, the coherent sheaf $R^a \oint_* (\mathcal{E}_N^{\mathrm{can}}(n\mathsf{D}')/\mathcal{E}_N^{\mathrm{can}}((n-1)\mathsf{D}'))$ is generated by its global sections, and its higher cohomology groups all vanish, over $\mathsf{M}^{\min}$. Consequently, by the Leray spectral sequence [13, Ch. II, Thm. 4.17.1], it suffices to show that, for some integer $N \geq N_0$ and all $a < c_{\mathsf{M}} - 1$, we have

$$H^a(\mathsf{M}^{\mathrm{tor}}, \mathcal{E}_N^{\mathrm{can}}(n'\mathsf{D}')/\mathcal{E}_N^{\mathrm{can}}(n\mathsf{D}')) = 0.$$

This is just (8.3) in Theorem 8.1, with $\mathcal{E}$ replaced with $\mathcal{E}_N$. □

Lemma 8.6. *In order to prove Theorem* 8.1*, it suffices to show that, in the case where R is an algebraically closed field, for each automorphic bundle $\mathcal{E}' = \mathcal{E}(W')$ over M with subcanonical extension $(\mathcal{E}')^{\mathrm{sub}} = \mathcal{E}^{\mathrm{sub}}(W')$ over $\mathsf{M}^{\mathrm{tor}}$ (both associated with some $W' \in \operatorname{Rep}_R(\mathrm{M}_0)$), for all integers $n' \geq n \geq 0$, and for all $a > \dim(\mathsf{M}^{\min} - \mathsf{M}) + 1$ (resp. $a = \dim(\mathsf{M}^{\min} - \mathsf{M}) + 1$), the canonical morphism*

$$H^a(\mathsf{M}^{\mathrm{tor}}, (\mathcal{E}')^{\mathrm{sub}}(-n'\mathsf{D}')) \to H^a(\mathsf{M}^{\mathrm{tor}}, (\mathcal{E}')^{\mathrm{sub}}(-n\mathsf{D}')) \tag{8.7}$$

is bijective (resp. surjective).

Proof. As explained in Section 3, we may assume that R is of finite type over R_0, and replace S_0 etc with their base changes to R. Since $\mathsf{M}^{\mathrm{tor}}$ is proper over S_0, and since $\mathcal{E}^{\mathrm{can}}(n'\mathsf{D}')/\mathcal{E}^{\mathrm{can}}(n\mathsf{D}')$ are flat over R_0 for all integers $n' \geq n$, by the usual base change arguments as in [38, Sec. 5] and [18, III, 7.7.5 and 7.7.10], in order to show that (8.3) holds for all $a < c_{\mathsf{M}} - 1$, it suffices to show the assertion after replacing R with its residue fields, or any of their field extensions. Then it suffices to show the equivalent assertion that (8.2) is bijective (resp. injective) for all $a < c_{\mathsf{M}} - 1$ (resp. $a = c_{\mathsf{M}} - 1$) under the assumption that R is an algebraically closed field.

By the extended Kodaira–Spencer isomorphism (see [32, Thm. 6.4.1.1(4)]), there exists some $W_{\mathcal{K}} \in \operatorname{Rep}_R(\mathrm{M}_0)$ such that the canonical bundle $\mathcal{K}$ over $\mathsf{M}^{\mathrm{tor}}$ is isomorphic to $\mathcal{E}^{\mathrm{sub}}(W_{\mathcal{K}})$. Thus, for $W' := \operatorname{Hom}_R(W, W_{\mathcal{K}})$ and $(\mathcal{E}')^{\mathrm{sub}} := \mathcal{E}^{\mathrm{sub}}(W')$, we have $\underline{\operatorname{Hom}}_{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}}(\mathcal{E}^{\mathrm{can}}(n\mathsf{D}'), \mathcal{K}) \cong (\mathcal{E}')^{\mathrm{sub}}(-n\mathsf{D}')$ for all $n \in \mathbb{Z}$. Then the bijectivity (resp. injectivity) of (8.2) for $a < c_{\mathsf{M}} - 1$ (resp. $a = c_{\mathsf{M}} - 1$) is equivalent to the bijectivity (resp. surjectivity) of (8.7) for $a > \dim(\mathsf{M}^{\min} - \mathsf{M}) + 1$ (resp. $a = \dim(\mathsf{M}^{\min} - \mathsf{M}) + 1$) by Serre duality (see, for example, [24, Cor. 7.7 and 7.12]), because either $\mathsf{M}^{\min} - \mathsf{M} \neq \emptyset$ and $\dim(\mathsf{M}^{\mathrm{tor}}) - c_{\mathsf{M}} = \dim(\mathsf{M}^{\min}) - c_{\mathsf{M}} = \dim(\mathsf{M}^{\min} - \mathsf{M})$; or $\mathsf{M}^{\min} - \mathsf{M} = \emptyset$ and (8.7) is bijective for all a. □

Remark 8.8. Because the proof of Lemma 8.4 (resp. Lemma 8.6) uses Serre's fundamental theorem for projective schemes (resp. Serre duality), it requires $\mathsf{M}^{\min}$ (resp. $\mathsf{M}^{\mathrm{tor}}$) to be projective (resp. projective and smooth) over S_0. (In particular, it does not work for partial compactifications as in [28].)

Proof of Theorem 8.1. By Lemma 8.6, we may assume that R is a field. By Theorem 3.9, for all integers $n' \geq n \geq 0$, the canonical short exact sequence

$$\begin{aligned} 0 &\to (\mathcal{E}')^{\mathrm{sub}}(-n'\mathsf{D}') \to (\mathcal{E}')^{\mathrm{sub}}(-n\mathsf{D}') \\ &\to \mathcal{Q} := ((\mathcal{E}')^{\mathrm{sub}}(-n\mathsf{D}'))/((\mathcal{E}')^{\mathrm{sub}}(-n'\mathsf{D}')) \to 0 \end{aligned}$$

induces a canonical short exact sequence

$$\begin{aligned} 0 &\to \oint_*((\mathcal{E}')^{\mathrm{sub}}(-n'\mathsf{D}')) \to \oint_*((\mathcal{E}')^{\mathrm{sub}}(-n\mathsf{D}')) \to \oint_* \mathcal{Q} \\ &\to R^1 \oint_*((\mathcal{E}')^{\mathrm{sub}}(-n'\mathsf{D}')) = 0. \end{aligned} \tag{8.9}$$

By Theorem 3.9 again, and by the Leray spectral sequence [13, Ch. II, Thm. 4.17.1], the morphism (8.7) can be identified with the canonical morphism

$$H^a(\mathsf{M}^{\min}, \oint_*((\mathcal{E}')^{\mathrm{sub}}(-n'\mathsf{D}'))) \to H^a(\mathsf{M}^{\min}, \oint_*((\mathcal{E}')^{\mathrm{sub}}(-n\mathsf{D}')))), \tag{8.10}$$

which fits into the long exact sequence

$$\begin{aligned} \cdots &\to H^{a-1}(\mathsf{M}^{\min}, \oint_* \mathcal{Q}) \to H^a(\mathsf{M}^{\min}, \oint_*((\mathcal{E}')^{\mathrm{sub}}(-n'\mathsf{D}'))) \\ &\to H^a(\mathsf{M}^{\min}, \oint_*((\mathcal{E}')^{\mathrm{sub}}(-n\mathsf{D}')))) \to H^a(\mathsf{M}^{\min}, \oint_* \mathcal{Q}) \to \cdots \end{aligned}$$

induced by (8.9). Since the sheaf $\mathcal{Q}$ is supported on $\mathsf{M}^{\mathrm{tor}} - \mathsf{M}$ because D' is, the sheaf $\oint_* \mathcal{Q}$ is supported on $\mathsf{M}^{\min} - \mathsf{M} = \oint(\mathsf{M}^{\mathrm{tor}} - \mathsf{M})$, and so $H^b(\mathsf{M}^{\min}, \oint_* \mathcal{Q}) = 0$ for $b > \dim(\mathsf{M}^{\min} - \mathsf{M})$ (see [15, Thm. 3.6.5] or [13, Ch. II, 4.15.2]). Consequently, (8.10) is bijective (resp. surjective) for $a > \dim(\mathsf{M}^{\min} - \mathsf{M}) + 1$ (resp. $a = \dim(\mathsf{M}^{\min} - \mathsf{M}) + 1$), and Theorem 8.1 follows, by Lemma 8.6. □

By Lemma 8.4, the proof of Theorem 2.5 is now complete.

Remark 8.11. In the proof of Lemma 8.4, it suffices to assume that there exists some integer $n_0 > 0$ (depending on $\mathcal{E}$) such that Theorem 8.1 holds for all integers $n' \geq n \geq 0$ divisible by n_0. Hence, for proving Theorem 2.5, we only need the existence of some integer $n_0 > 0$ (depending on $\mathcal{E}'$) such that $R^a \oint_*(\mathcal{E}')^{\mathrm{sub}}(-n\mathsf{D}') = 0$ for all $a > 0$ and all integers $n \geq 0$ divisible by n_0. (This is a special case of Theorem 3.9.) While the case $n = 0$ is the most essential, the existence of some $n_0 > 0$ such that $R^a \oint_*(\mathcal{E}')^{\mathrm{sub}}(-n\mathsf{D}') = 0$ for all $a > 0$ and all integers $n > 0$ divisible by n_0 follows from Serre's fundamental theorem [18, III, 2.2.1] and from the relative ampleness of $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}}(-\mathsf{D}')$ over $\mathsf{M}^{\min}$ (see [4, Ch. IV, Sec. 2.1, Thm. 2.2], [10, Ch. V, Thm. 5.8], and [32, Thm. 7.3.3.4]; see also [34, property (5) preceding (2.1)]).

Remark 8.12. In the setup of complex-analytically constructed compactifications as in [3] etc, by [39, Prop. 3.4 b)], the log canonical bundle $\mathcal{K}(\log \mathsf{D})$ over $\mathsf{M}^{\mathrm{tor}}$, which is isomorphic to $\mathcal{E}^{\mathrm{can}}(W_{\mathcal{K}})$ for some $W_{\mathcal{K}} \in \mathrm{Rep}_R(\mathrm{M}_0)$ (cf. the proof of Lemma 8.6), descends to an ample invertible sheaf over $\mathsf{M}^{\min}$. This ample invertible sheaf over $\mathsf{M}^{\min}$ can serve the same purpose of $\omega^{\min}$ in all our arguments in this article.

9. Failure in degree equal to codimension minus one

For simplicity, we shall assume in this section that R is of finite type over R_0 and *Cohen–Macaulay*. Also, we shall assume that $\mathsf{M}^{\min} - \mathsf{M} \neq \emptyset$; otherwise the assertions are vacuous. As in Section 3, we shall replace S_0, M, $\mathsf{M}^{\mathrm{tor}}$, $\mathsf{M}^{\min}$, etc with their base changes from $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ to $\mathrm{Spec}(R)$.

Proposition 9.1 (cf. the first paragraph of Theorem 2.5)**.** *The canonical morphism* (2.6) *is not an isomorphism for* $a = c_{\mathsf{M}} - 1$.

Proof. Since $\mathsf{M}^{\min}$ is of finite type over $\mathsf{M}_0 = \mathrm{Spec}(R)$, and since R is of finite type over R_0 (by assumption), $\mathsf{M}^{\min}$ is of finite type over the regular scheme $\mathrm{Spec}(R_0)$, and hence can be locally embedded in a regular scheme. Since M is smooth over $\mathsf{M}_0 = \mathrm{Spec}(R)$, and since R is Cohen–Macaulay by assumption, by [18, IV-4, 17.5.8], the locally free sheaf $\mathcal{E}$ over M is Cohen–Macaulay. Hence, by [16, VIII, Prop. 3.2], $R^a j^{\min}_* \mathcal{E}$ cannot be coherent over $\mathsf{M}^{\min}$ for $a = c_{\mathsf{M}} - 1$. Since $R^a \oint_* (\mathcal{E}^{\mathrm{can}})$ is coherent over $\mathsf{M}^{\min}$ for all a, it follows that (2.6) cannot be an isomorphism for $a = c_{\mathsf{M}} - 1$. □

Proposition 9.2 (cf. the second paragraph of Theorem 2.5)**.** *There exists some integer N_0 such that, for every integer $N \geq N_0$, the canonical morphism*

$$H^{c_{\mathsf{M}}-1}(\mathsf{M}^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}_N) \to H^{c_{\mathsf{M}}-1}(\mathsf{M}, \mathcal{E}_N) \tag{9.3}$$

is not bijective, where $\mathcal{E}_N$ and $\mathcal{E}^{\mathrm{can}}_N$ are as in the paragraph preceding Lemma 8.4*.* (*Hence Theorem* 2.5 *is sharp up to replacing $\mathcal{E}$ and $\mathcal{E}^{\mathrm{can}}$ with their tensor products with pullbacks of sufficiently high powers of $\omega^{\min}$.*)

Proof. By Proposition 9.1 and (3.2), there exists some integer $n_1 > 0$ such that (3.3) is not an isomorphism for $n = n_1$ and $a = c_{\mathsf{M}} - 1$. By the projection formula [18, 0_{I}, 5.4.10.1], for every $N \in \mathbb{Z}$, the canonical morphism

$$R^{c_{\mathsf{M}}-1} \oint_* (\mathcal{E}^{\mathrm{can}}_N) \to R^{c_{\mathsf{M}}-1} \oint_* (\mathcal{E}^{\mathrm{can}}_N(n_1 \mathsf{D}'))$$

is not an isomorphism. Since $\omega^{\min}$ is ample over $\mathsf{M}^{\min}$, by the same argument as in the proof of Lemma 8.4, there exists some integer N_0 such that, for every integer $N \geq N_0$, the canonical morphism

$$H^{c_{\mathsf{M}}-1}(\mathsf{M}^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}_N) \to H^{c_{\mathsf{M}}-1}(\mathsf{M}^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}_N(n_1 \mathsf{D}'))$$

is not bijective. By Theorem 8.1, the canonical morphisms as in (8.2), with $\mathcal{E}$ replaced with $\mathcal{E}_N$, are injective for all $n' \geq n \geq 0$ and for $a = c_{\mathsf{M}}$. Moreover, by (3.1) and the Leray spectral sequence [13, Ch. II, Thm. 4.17.1],

$$H^a(\mathsf{M}, \mathcal{E}_N) \cong \varinjlim_{n \geq 0} H^a(\mathsf{M}^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}_N(n \mathsf{D}')). \tag{9.4}$$

Thus, for every integer $N \geq N_0$, (9.3) is not bijective. □

10. Remark on other cases

The methods presented for the PEL-type setup in [32] and [31] also work in several other setups. Let us record the explanations as concluding remarks.

Remark 10.1. The methods (for proving all the results) also work for the complex-analytically constructed compactifications of all Shimura varieties (or disjoint unions of their connected components) as in [3], [4], [21], and [41]. This is because, even in non-PEL-type cases, we still have compatible proper morphisms to the minimal compactifications from the toroidal compactifications associated with projective and smooth cone decompositions, with exactly the same description of (formal) local structures along the fibers (of the proper morphisms), apart from some notational differences. (In the PEL-type case, see the explicit comparison of formal charts in [30].) In all steps of our proofs, we have provided arguments (or references for them) that also work in non-PEL-type cases.

Remark 10.2. The methods for proving Theorems 2.3 and 3.9 in Sections 4, 5, 6, and 7 also work for the partial compactifications of ordinary loci as in [28], over arbitrarily ramified base rings. This is because we still have compatible proper morphisms to the partial minimal compactifications from the partial toroidal compactifications associated with projective and smooth cone decompositions, with exactly the same description of (formal) local structures along the fibers (of the proper morphisms). In fact, when we proved Theorem 3.9 in the special case $n = 0$ in [28, Sec. 8.2] (see

Remark 8.11), we spelled out the argument for such partial compactifications of ordinary loci (and omitted the proof for the proper smooth good reduction models), and we had essentially the same setup. However, as explained in Remark 8.8, the methods for deducing Theorems 2.5 and 8.1 from Theorem 3.9 by duality, and for proving Proposition 9.2, do not work for such partial compactifications.

Remark 10.3. The methods for proving Theorems 2.3 and 3.9 in Sections 4, 5, 6, and 7 should also work for other integral models of toroidal and minimal compactifications having exactly the same description of (formal) local structures along the fibers (of the proper morphisms), such as the p-integral models of toroidal and minimal compactifications of Hodge-type Shimura varieties at levels maximal hyperspecial at p constructed in [37].

Remark 10.4. Our results naturally extend to the case of automorphic bundles associated with the algebraic group scheme P_0 as in [31, Sec. 6]. More precisely, as explained in [35, Cor. 1.21, Lem. 2.11, and Def. 2.12] and [36, Lem. 4.14 and Def. 4.19], each $W \in \mathrm{Rep}_R(\mathrm{P}_0)$ has a "Hodge filtration" with graded pieces given by objects W_i in $\mathrm{Rep}_R(\mathrm{M}_0)$, and the associated $\mathcal{E}_{\mathrm{P}_0,R}(W)$, $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0,R}(W)$, and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_0,R}(W)$ admit filtrations with graded pieces given by $\mathcal{E}_{\mathrm{M}_0,R}(W_i)$, $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,R}(W_i)$, and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{M}_0,R}(W_i)$, respectively. By the spectral sequences associated with such filtrations, the comparison and vanishing results we have proved for the automorphic bundles associated with M_0 are also true for the automorphic bundles associated with P_0. Similarly, under the assumption that [28, Cond. 8.1.1.2] holds, the analogues of Theorems 2.3 and 3.9 for the automorphic bundles associated with $\mathrm{P}^{\mathrm{ord}}_{\mathrm{D},0}$ also hold over the ordinary loci (see Remark 10.2). When $R = \mathbb{C}$, while the global sections of $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,R}(W_i)$ are represented by holomorphic automorphic forms, the global sections of $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0,R}(W)$ are represented by the so-called *nearly holomorphic automorphic forms*. (See [30, Thm. 5.2.12] for the comparison between the algebraic and analytic constructions of $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0,R}(W)$.) The nearly holomorphic automorphic forms were first introduced by Shimura in the modular curve case (see [46]), and later studied in general by Harris and others (see [19], [20], and [40]; see also [49] and its introduction for a summary on the current literature). Thus we have also obtained a *nearly holomorphic* Koecher's principle (for all PEL-type cases in good mixed characteristics; or for all complex-analytically constructed cases as explained in Remark 10.1).

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